STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 26

Unit 8 : Gauss’ Law; Equation of Continuity

Essential tools to study Fluid Mechanics / Electrodynamics, etc.

Previous Unit: Unit 7

Physical examples of fields.

Potential energy function.

Gradient, Directional Derivative, ….

Mathematical

Connections…..

$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$$

Potentials —— **Differentiation** ——→ Fields

←——

Integration / Constant of Integration

Boundary Value problem

SCALAR
POINT FUNCTIONS

VECTOR
POINT FUNCTIONS

Gauss' Law;
Equation of Continuity.
Hydrodynamics &
Electrodynamics
illustrations.

Johann
Carl
Friedrich
Gauss
1777 - 1855

Learning goals:

When there is no source and no sink, the density of matter in a volume element can change if and only of matter flows in, *or out*, of that region across the surface that bounds that volume region.

The divergence theorem : an exact mathematical expression of a conservation principle.

The result is equally consequential with regard to fields just as well as for matter.

We shall develop further handle on methods of vector calculus and apply the techniques to study fluid dynamics and electrodynamics.

Johann Carl Friedrich Gauss and Wilhelm Weber

http://www.gap-system.org/~history/PictDisplay/Gauss.html

$$\underset{\text{volume region}}{\iiint} dV\left[\vec{\nabla}\bullet\vec{A}(\vec{r})\right] = \underset{\text{surface enclosing that region}}{\iint} \vec{A}(\vec{r})\bullet dS\hat{n}$$

Recapitulate From Unit 7

Consolidated expressions for the GRADIENT

Cartesian Coordinate System

$$\vec{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$$

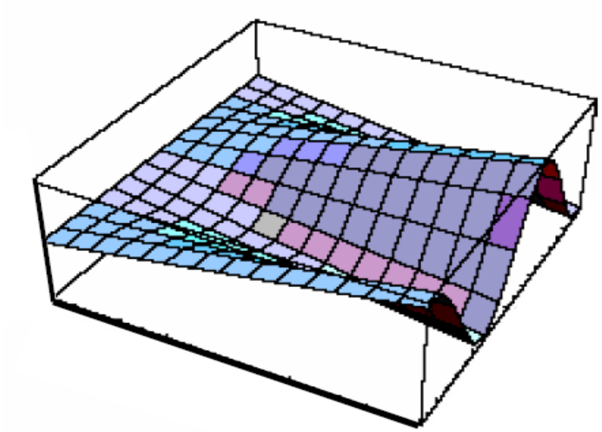
$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\vec{\delta r}}{\delta s} = \frac{\vec{dr}}{ds}$$

$$\delta s = \left|\vec{\delta r}\right|$$

Cylindrical Polar Coordinate System

$$\vec{\nabla} = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}$$



Spherical Polar Coordinate System

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

The 'GRADIENT' is a vector operator

– it is of course not a vector.

The operator would operate on an operand and generate new entities as a result of the operation.

$$Operand : SCALAR\ POINT\ FUNCTION.\ \ RESULT : \vec{\nabla}\psi$$

$$Other\ operations\ \text{using}\ \ GRADIENT\ OPERATOR\ \ \vec{\nabla}$$

$\vec{\nabla} \cdot \vec{A}(\vec{r})$: DIVERGENCE of a VECTOR POINT FUNCTION

$\vec{\nabla} \times \vec{A}(\vec{r})$: CURL of a VECTOR POINT FUNCTION

$GRADIENT$ of $SCALAR\ POINT\ FUNCTION.\ RESULT: \vec{\nabla}\psi$

$Other\ operations$ using $GRADIENT$ $OPERATOR$ $\vec{\nabla}$

$\vec{\nabla}\cdot\vec{A}(\vec{r})$: DIVERGENCE of a VECTOR POINT FUNCTION

This is NOT a scalar product of two vectors!

$\vec{\nabla}\times\vec{A}(\vec{r})$: CURL of a VECTOR POINT FUNCTION

This is NOT a vector product of two vectors!

Recapitulate Scalar/Vector Fields; 'point functions'

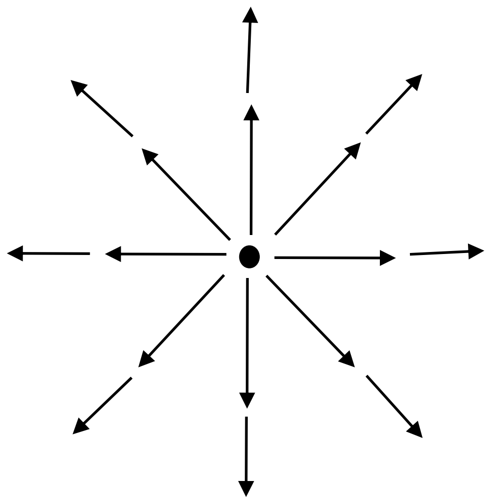
Field Lines

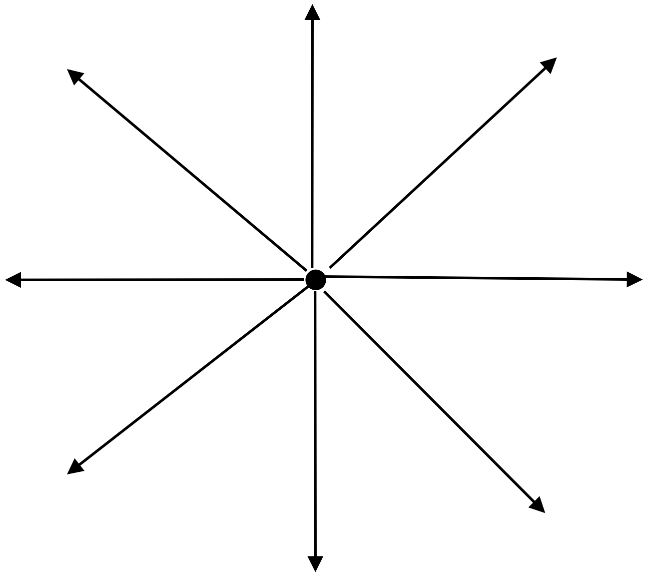
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{e}_r$$

Field strength:

VECTOR POINT FUNCTION

Field intensity fall like $1/r^2$





Vector Fields: 'Point' function

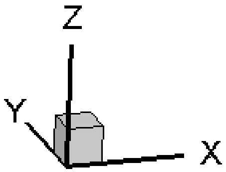
$$\vec{V} = \vec{V}(\vec{r}) = \vec{V}(x, y, z)$$

$$= \vec{V}(r, \theta, \varphi) = \vec{V}(\rho, \varphi, z)$$

$$\vec{V} = \vec{V}(\vec{r}, t)$$

discrete
versus
continuous

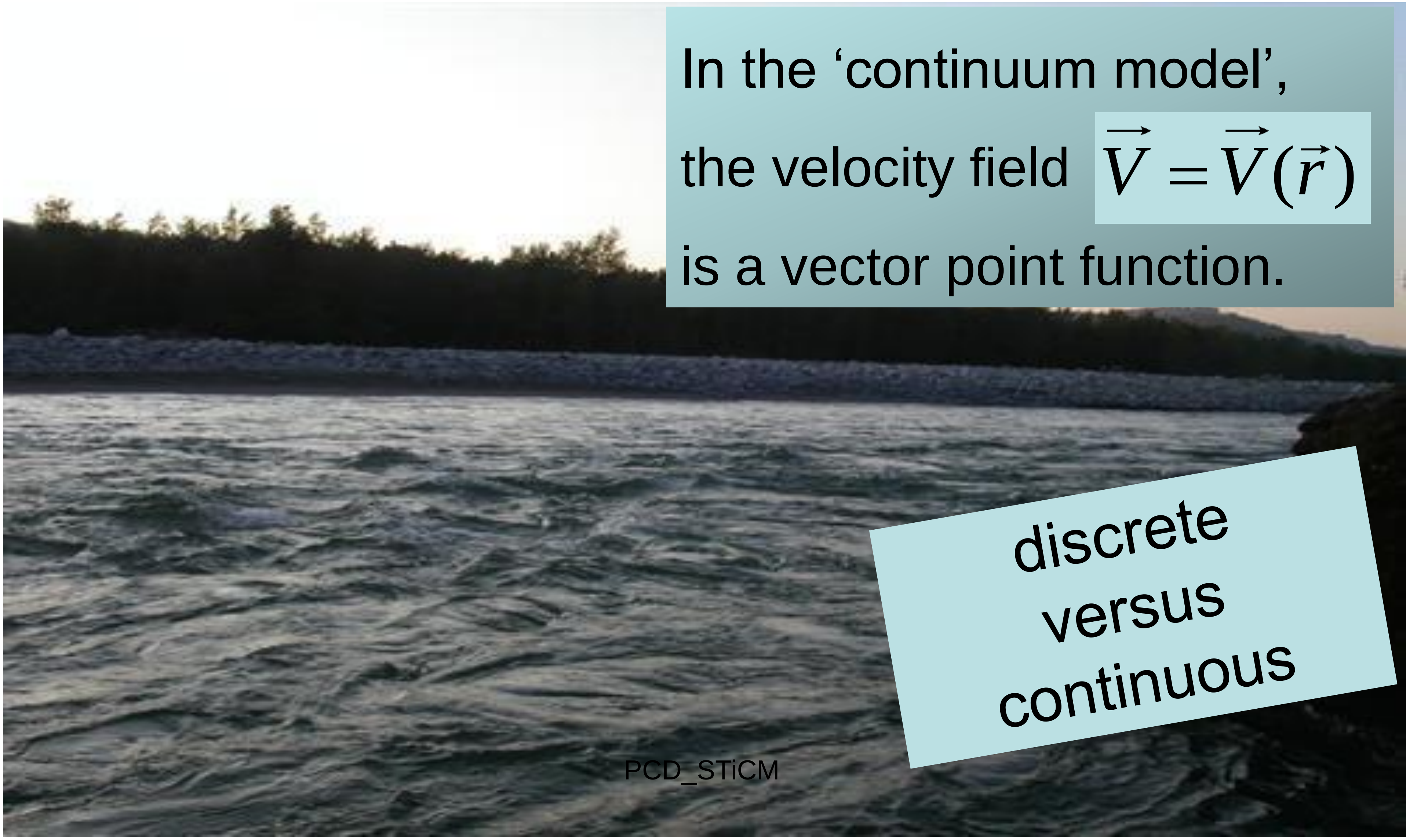
Vector Fields: 'Point' function



$$\vec{V} = \vec{V}(\vec{r}) = \vec{V}(x, y, z)$$

$$= \vec{V}(r, \theta, \varphi) = \vec{V}(\rho, \varphi, z)$$

$$\vec{V} = \vec{V}(\vec{r}, t)$$



In the 'continuum model', the velocity field $\vec{V} = \vec{V}(\vec{r})$ is a vector point function.

discrete versus continuous

$\vec{A}(\vec{r})$: A vector point function.

Define: "Flux" of a vector point function

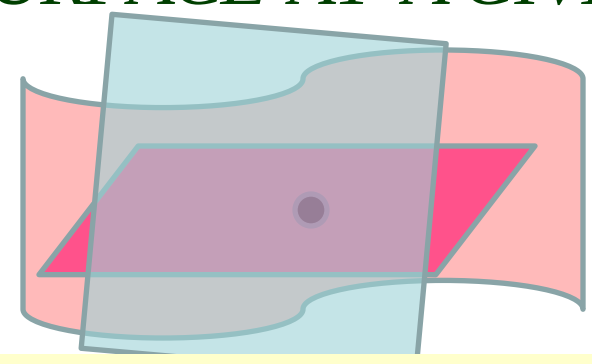
$$\text{Flux crossing a } \boxed{\text{surface}} = \boxed{\iint_{\text{surface}}} \vec{A}(\vec{r}) \bullet \boxed{dS\hat{n}}$$

Flux: additive property - obtained by integrating the quantity

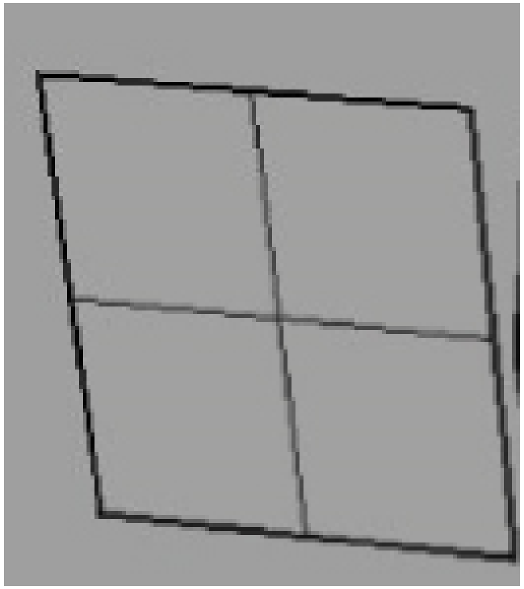
what is the direction/orientation of $\hat{n}$?

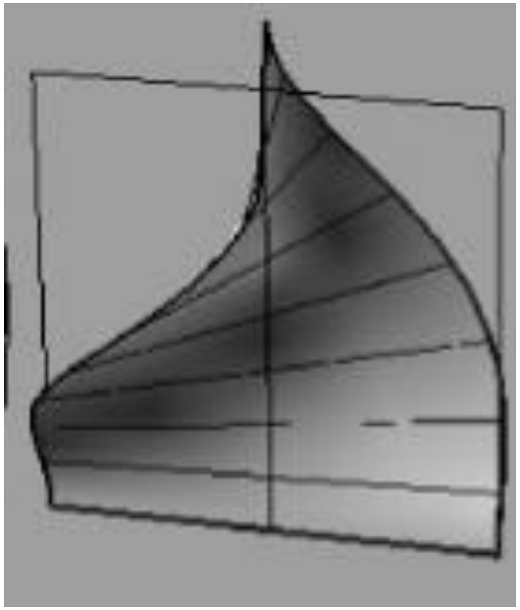
UNIT NORMAL TO THE SURFACE AT A GIVEN POINT

.... but *which* surface?



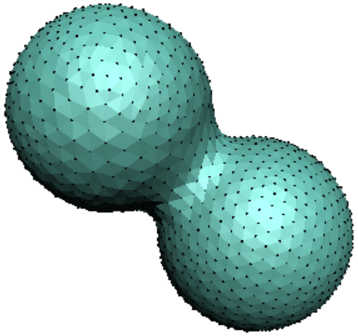
.. what is the direction of the unit normal?





$\vec{A}(\vec{r})$:

A vector point function.



$$\text{Flux crossing a surface} = \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Elemental

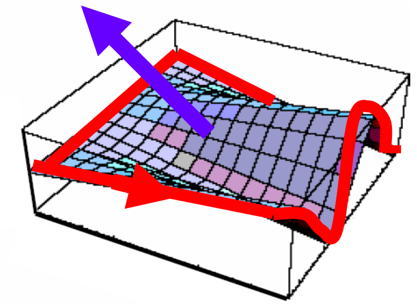
directed/oriented

'Area'

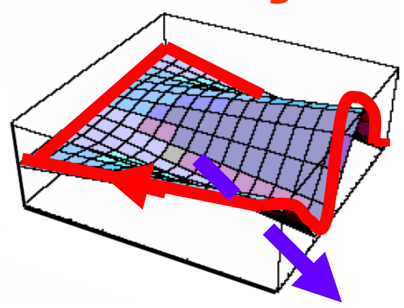
Flux crossing a surface

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

The direction of the vector surface element must be defined in a manner that is consistent with the forward-movement of a right-hand screw.



C traversed one way



C traversed the other way

right-hand-screw convention must be followed.

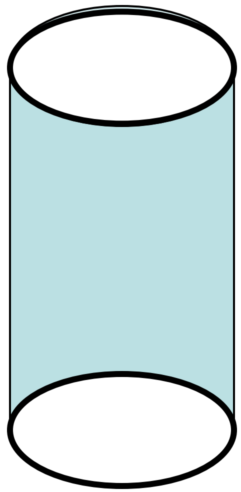
Consider the sense/direction in which the rim of the net can be traversed

… right hand screw

What about some other point ?

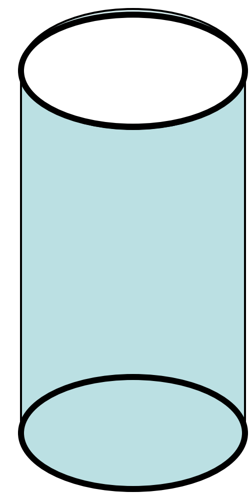
… and what if you ‘pinched’ the net and pulled it ‘above’ the rim?

Are there non-orientable surfaces?



The surface under consideration, however, better be a ‘well-behaved’ surface!

A cylinder open at both ends is *not* a ‘well-behaved’ surface!



A cylinder open at only one end is ‘well-behaved’; isn’t it already like the butterfly net?

Consider a rectangular strip of paper, spread flat at first, and given two colors on opposite sides.

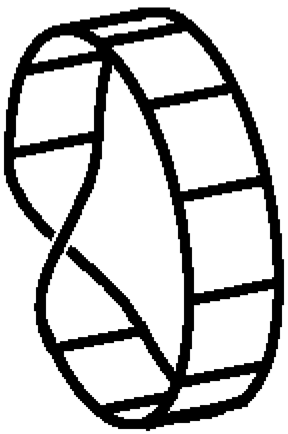
Now, flip it and paste the short edges on each other as shown.

Is the resulting object three-dimensional?

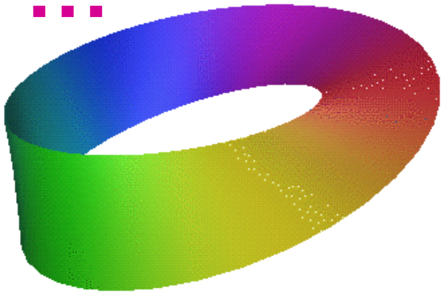
How many ***‘edges’*** does it have?

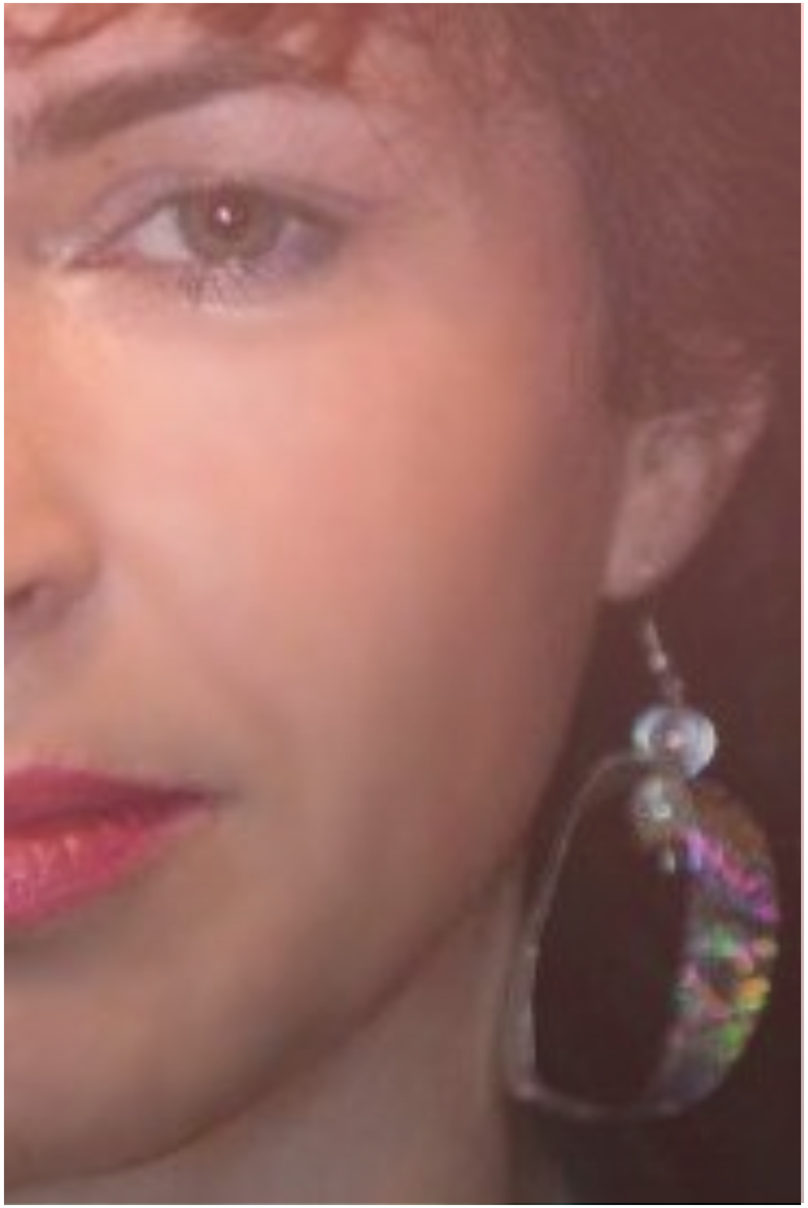
How many *‘sides’* does it have?

How is the earring?



It is MOBIUS !!!

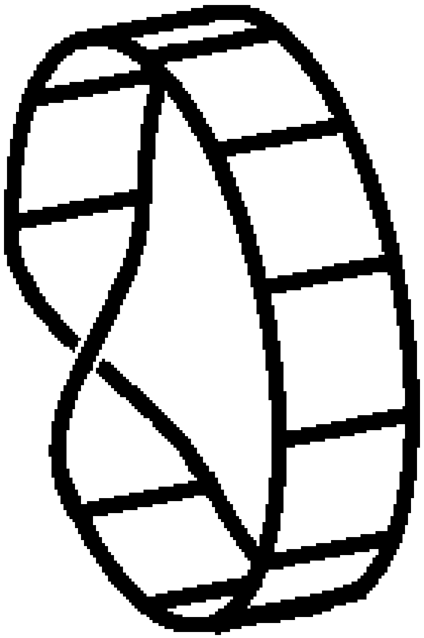


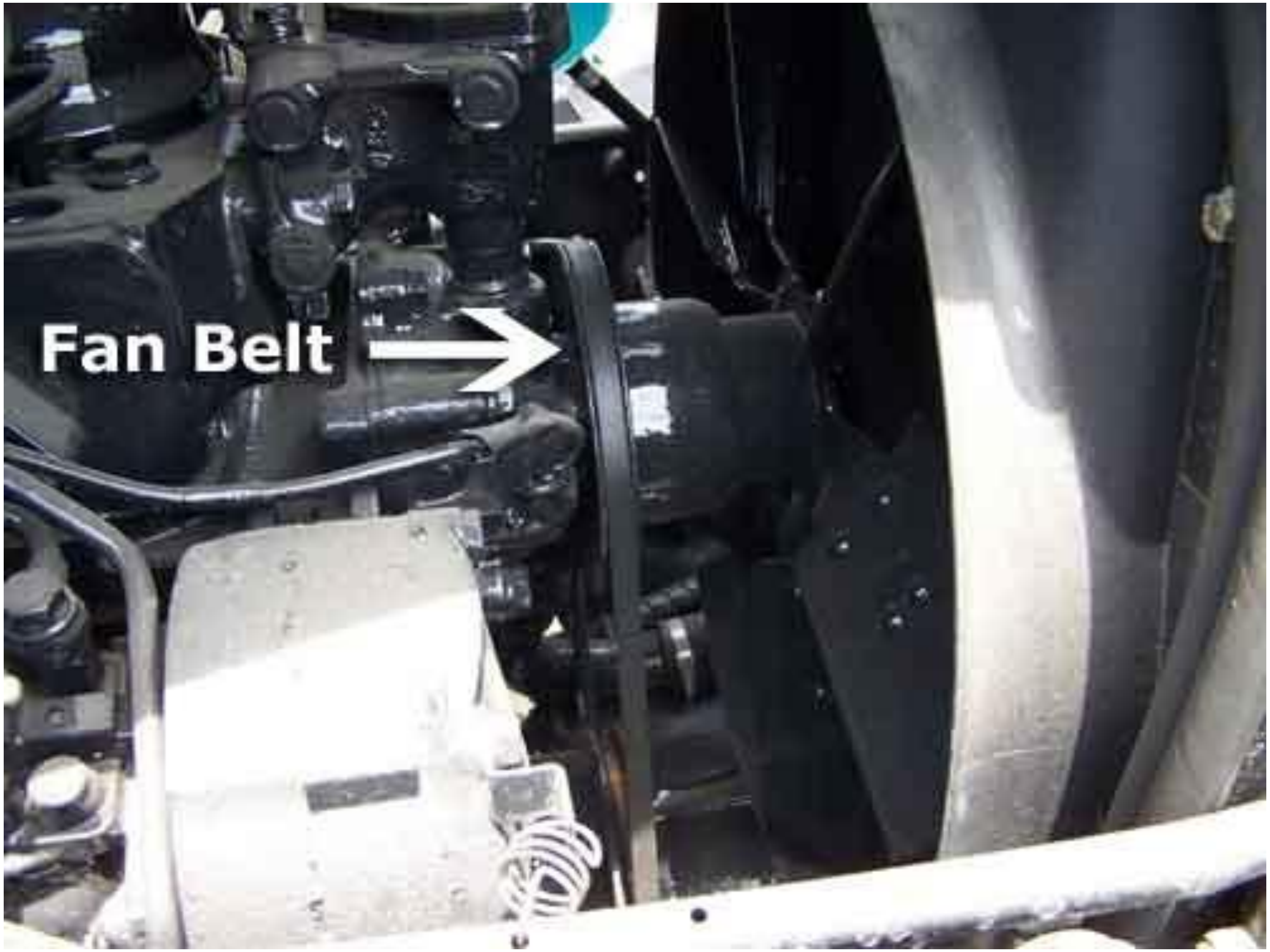


http://mathssquad.questacon.edu.au/mobius_strip.html

The Möbius strip used to be common in belt drives

(eg. car fan belt).





Modern belts are made from several layers of different materials, with a definite inside and outside, and do not have a twist.

Maurits
Cornelius
Escher
1898 - 1972

http://www.mcescher.com/

“The laws of mathematics are not merely human inventions or creations. They simply 'are'; they exist quite independently of the human intellect. The most that any(one) ... can do is to find that they are there, and to take cognizance of them. ”

Flux of a vector field

$\hat{n}(\vec{r})$: unit vector

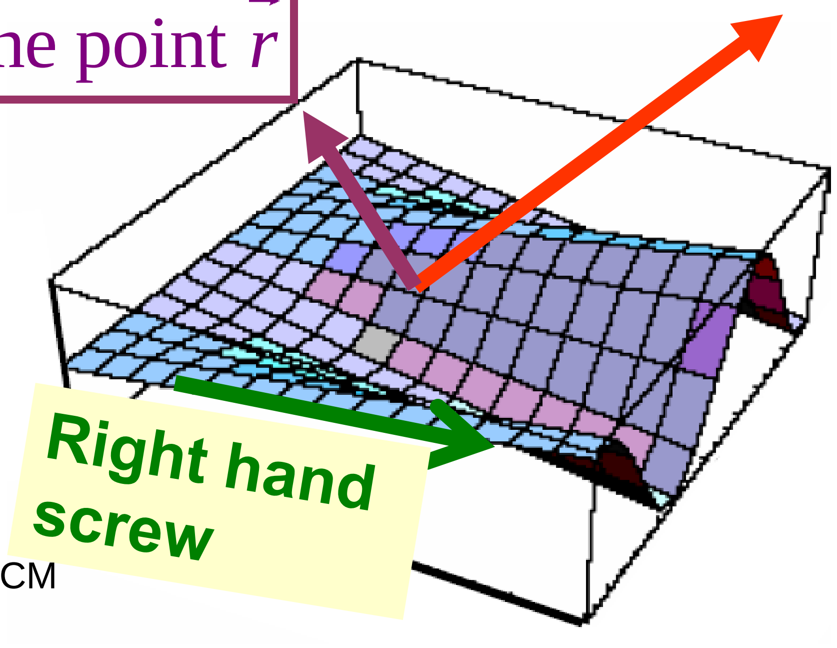
normal to the surface at the point $\vec{r}$

$\vec{A}(\vec{r})$: A vector

point function

Flux crossing a surface

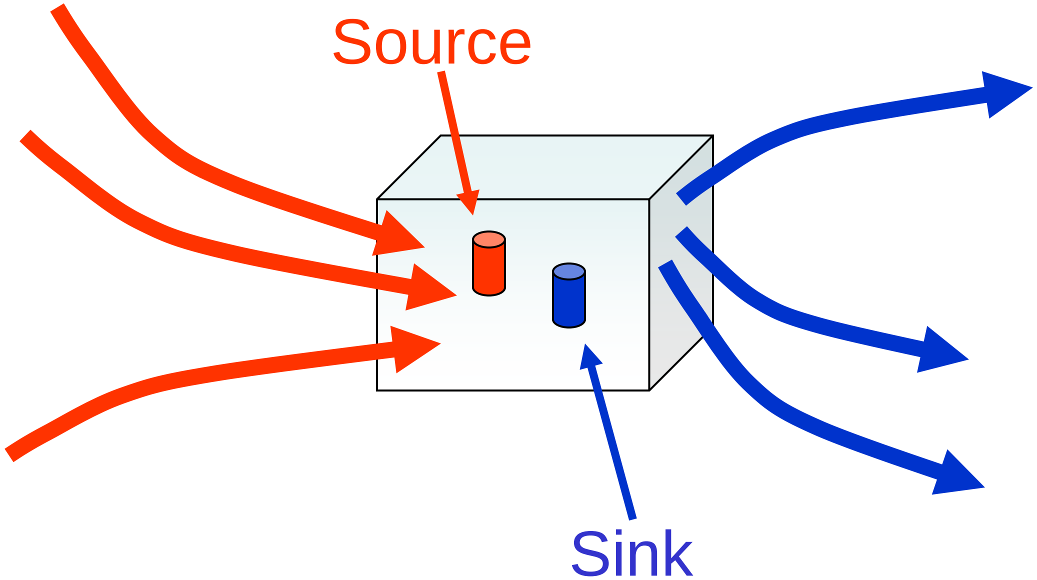
$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$



Flux crossing a surface

$$= \iint\limits_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Is there any net accumulation of the flux in a volume element?

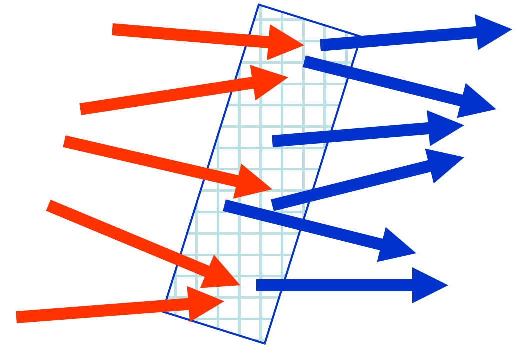


Sources and Sinks may be present in the region !

What happens when the size of the volume element shrinks,

becoming infinitesimally small?

Consider a mass/charge density ρ_m or ρ_c crossing a certain cross-section of area at a certain rate.



$$\rho_m(\vec{r})\vec{\mathrm{v}}(\vec{r}) \text{ has the dimensions}$$

$$\left[\mathrm{ML^{-3}\ LT^{-1}}\right] = \mathrm{ML^{-2}\ T^{-1}}$$

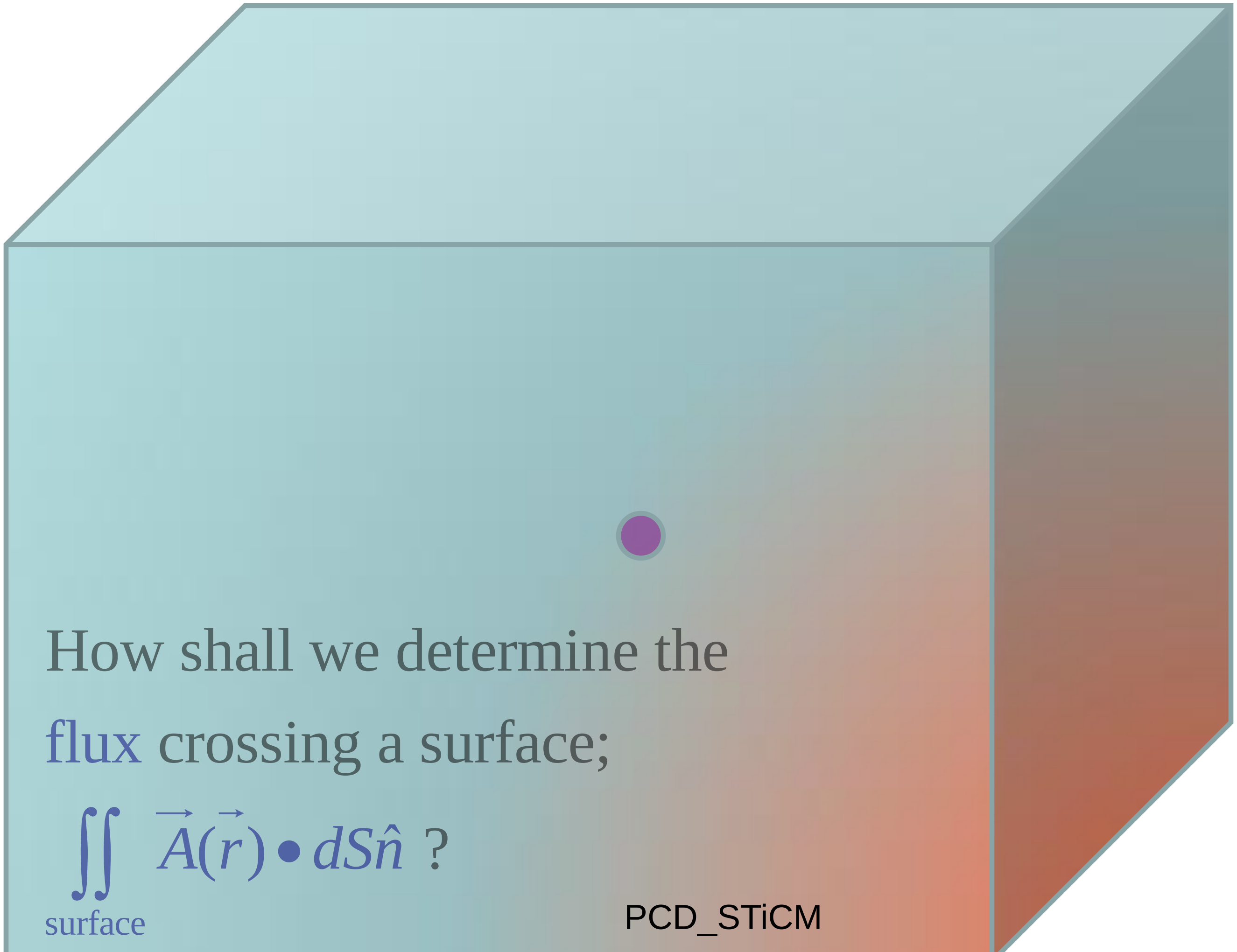
$\rho_c(\vec{r})\vec{\mathrm{v}}(\vec{r})$ has the dimensions $\left[\mathrm{QL^{-3}\ LT^{-1}}\right] = \mathrm{QL^{-2}\ T^{-1}}$

Amount of charge crossing unit area in unit time

Amount of mass/charge crossing unit area in unit time

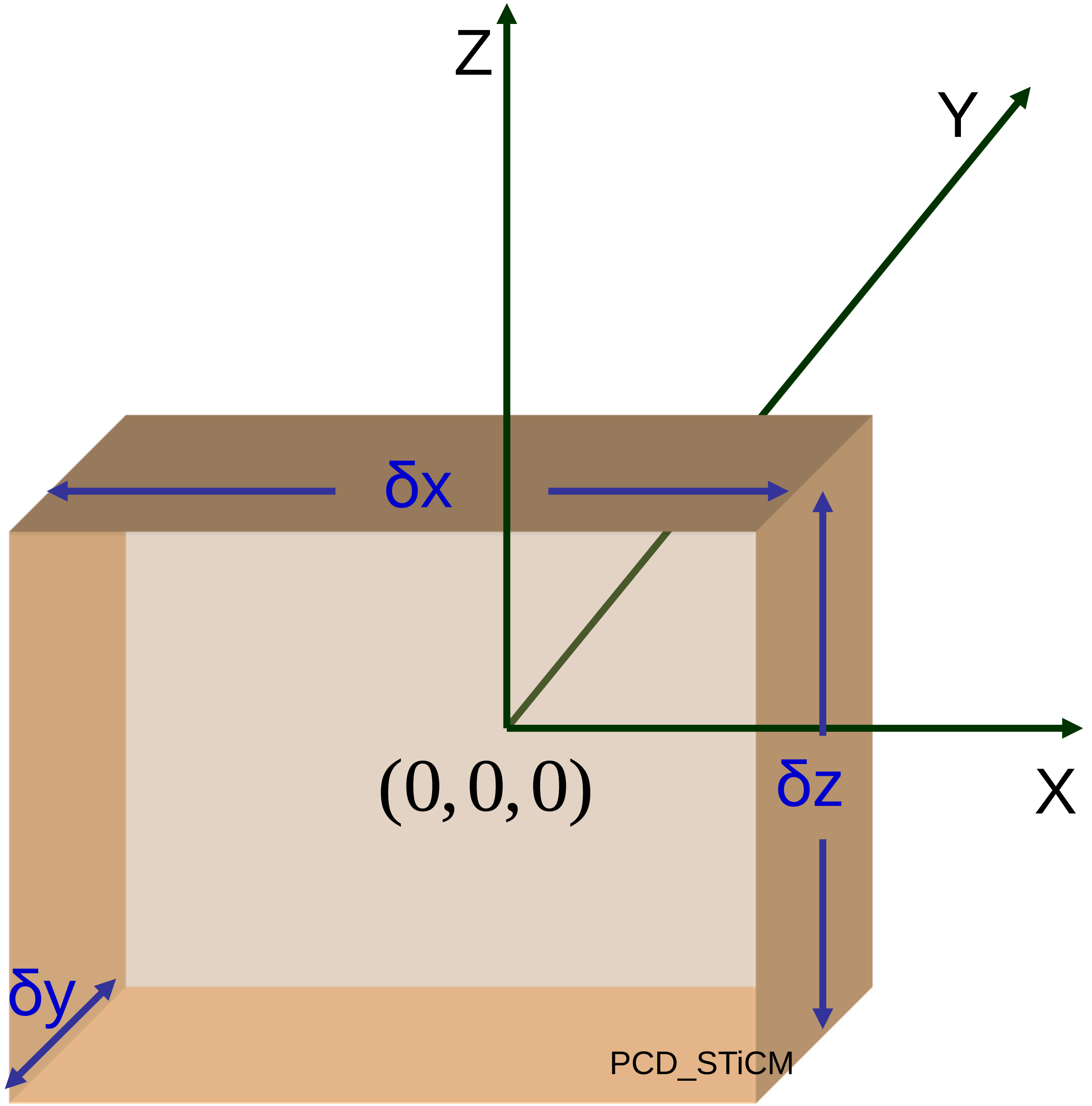
Consider a point $P(x_0, y_0, z_0)$

in a region of the vetor field $\vec{A}(\vec{r})$

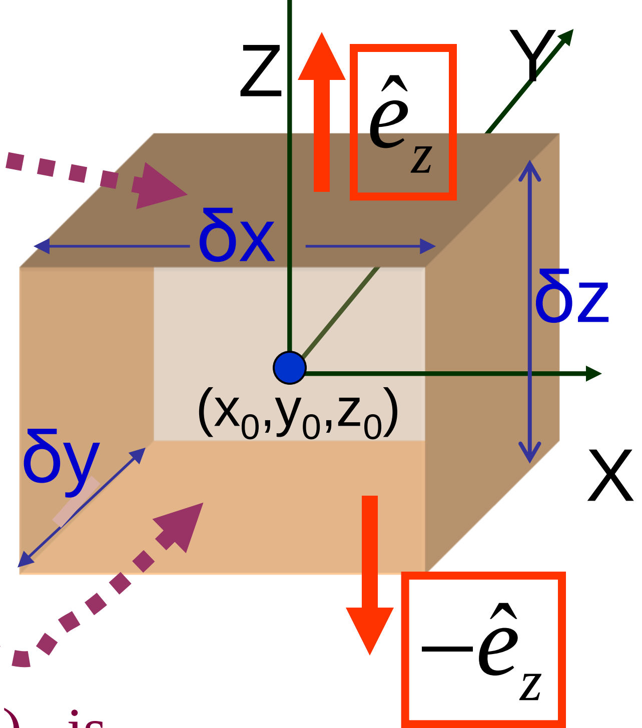


How shall we determine the flux crossing a surface;

$$\iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n} \ ?$$


Z
Y
δx
δz
X
(0, 0, 0)
δy

Two faces parallel to the xy plane



The point $P(x_0, y_0, z_0)$ is in the region of the vectore field $\vec{A}(\vec{r})$

NET Flux through the *xy*-face, (perpendicular to $\hat{e}_z$), is

$$\left[A_z\left(x_0, y_0, z_0 \oplus \frac{\delta z}{2}\right) - A_z\left(x_0, y_0, z_0 \ominus \frac{\delta z}{2}\right)\right]\delta x \delta y$$

Note! Only the z-component contributes to the total flux through the faces parallel to the xy plane.

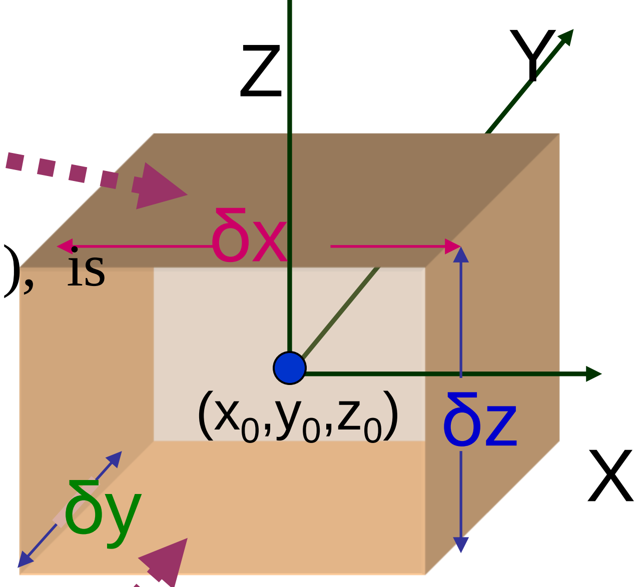
Flux crossing a surface

$$= \iint_{surface} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Flux crossing a surface

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

NET Flux through the *xy*-face, (perpendicular to $\hat{e}_z$), is



$$\left[A_z\left(x_0, y_0, z_0 + \frac{\delta z}{2} \right) - A_z\left(x_0, y_0, z_0 - \frac{\delta z}{2} \right) \right] \delta x \delta y$$

$$= \left[\frac{\partial A_z}{\partial z} \right]_{(x_0, y_0, z_0)} \delta z \delta x \delta y = \left[\frac{\partial A_z}{\partial z} \right]_{(x_0, y_0, z_0)} \delta V$$

Flux crossing all the six surface elements that enclose the cell

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Adding the flux through all faces, total flux

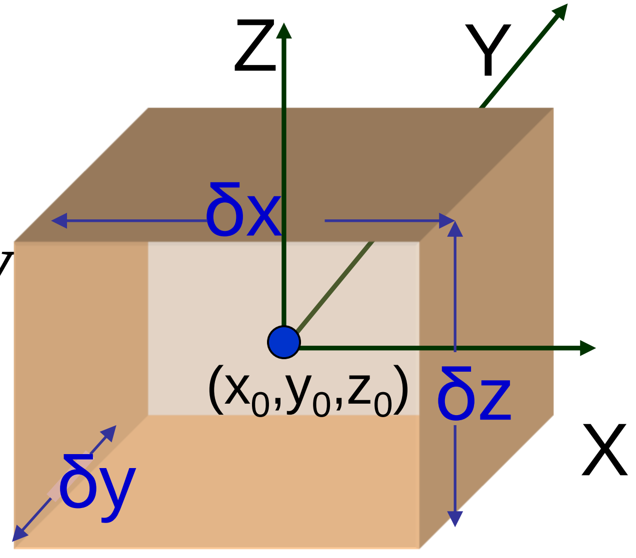
$$\oiint_{\textit{whole cube}} \vec{A}(\vec{r}) \cdot dS\hat{n} = \int \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] dV$$

Flux crossing a surface

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Adding the flux through all faces, total flux

$$\oiint_{\substack{closed\\ surface}} \vec{A}(\vec{r}) \bullet dS\hat{n} = \int \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] dV$$



The integrand of the volume integral is called the divergence of the vector.

$$\text{div}\,\vec{A} = \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] = \vec{\nabla} \bullet \vec{A}$$

$$\iiint_{\substack{\text{volume}\\ \text{region}}} d\tau \left[\vec{\nabla} \bullet \vec{A}(\vec{r}) \right] = \oiint_{\substack{\text{surface enclosing}\\ \text{that region}}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Gauss' Divergence Theorem

Divergence of a polar vector is a scalar

Divergence of an axial-vector is a pseudo-scalar

We shall take a break here…….

Questions ?

Comments ?

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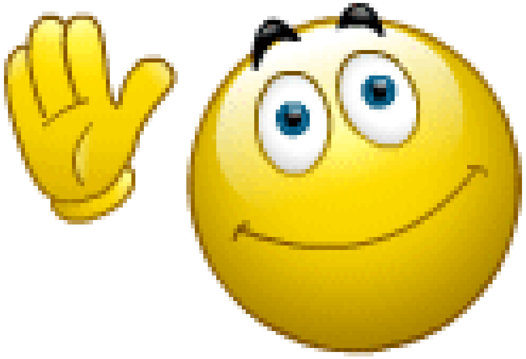
http://www.physics.iitm.ac.in/~labs/amp/

Next: L27

Equation of Continuity



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STiCM Lecture 27

Unit 8 : Gauss' Law; Equation of Continuity

Flux crossing a surface

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Adding the flux through all faces, total flux

$$\iiint_{\substack{\text{volume}\\ \text{region}}} d\tau \left[\vec{\nabla} \bullet \vec{A}(\vec{r}) \right] = \oiint_{\substack{\text{surface}\\ \text{enclosing}\\ \text{that}\\ \text{region}}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Gauss' Divergence Theorem

$$\underset{\text{volume region}}{\iiint} d\tau \left[\vec{\nabla} \bullet \vec{E}(\vec{r}) \right] = \underset{\text{surface enclosing that region}}{\oiint} \vec{E}(\vec{r}) \bullet dS\hat{n}$$

Application: electric intensity field due to a point charge

$$\underset{\text{volume region}}{\iiint} d\tau \left[\vec{\nabla} \bullet \vec{E}(\vec{r}) \right] = \underset{\text{surface enclosing that region}}{\oiint} \left(\frac{q}{4\pi\varepsilon_0 r^2} \hat{e}_r \right) \bullet \left(r^2 \sin\theta d\theta d\varphi \hat{e}_r \right)$$

$$\underset{\text{volume region}}{\iiint} d\tau \left[\vec{\nabla} \bullet \vec{E}(\vec{r}) \right] = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \underset{\text{volume region}}{\iiint} d\tau \rho = \underset{\text{volume region}}{\iiint} d\tau \frac{\rho}{\varepsilon_0}$$

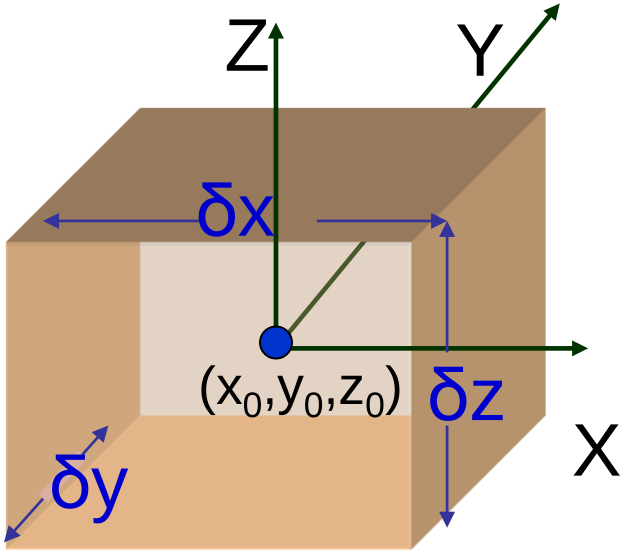
$$\vec{\nabla} \bullet \vec{E}(\vec{r}) = \frac{\rho}{\varepsilon_0}$$

Differential (or ‘point’) form of the Gauss’s law

Flux crossing a surface

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

$$\oint\!\!\oint_{\substack{closed\\ surface}} \vec{A}(\vec{r}) \bullet dS\hat{n} = \iiint \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] dV$$



The integrand of the volume integral is called the divergence of the vector.

$$\iiint_{\substack{\text{volume}\\ \text{region}}} dV \left[\vec{\nabla} \bullet \vec{A}(\vec{r}) \right] = \oint\!\!\oint_{\substack{\text{surface}\\ \text{enclosing}\\ \text{that}\\ \text{region}}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

Gauss' Divergence Theorem

$$\text{div}\, \vec{A} = \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] = \vec{\nabla} \bullet \vec{A}$$

Cartesian expression of 'divergence of the vector' not its definition!

Physical meaning of 'divergence'; definition free from coordinate system

$$\underset{\text{volume region}}{\iiint} dV\left[\vec{\nabla}\bullet\vec{A}(\vec{r})\right] = \underset{\text{surface enclosing that region}}{\iint} \vec{A}(\vec{r})\bullet dS\hat{n}$$

Take the limit of the ratio of total flux over δs to δV

$$\text{div}\vec{A} = \lim_{\delta V\to 0}\frac{\underset{\text{enclosing surface}}{\oiint}\vec{A}(\vec{r})\bullet\hat{n}dS}{\delta V} = \vec{\nabla}\bullet\vec{A}$$

flux per unit volume, at that point

remember: flux is defined through a SURFACE,
whereas divergence is defined at a *POINT*

Flux is a scalar quantity. It is not a scalar field; it is not a local quantity – it is not a 'point function'.

Divergence is a scalar field; it is a scalar point function, it is defined at each point of space

Gauss's Divergence Theorem

If a volume V is bounded by a surface S, then, for vector ***A*,**

The surface integral of the normal component of a vector $\vec{\mathbf{A}}$ taken over a closed surface is equal to the integral of the divergence of $\vec{\boldsymbol{A}}$ taken over the volume enclosed by the surface

$$\underset{\text{volume region}}{\iiint} dV\left[\vec{\nabla}\bullet\vec{A}(\vec{r})\right] = \underset{\text{surface enclosing that region}}{\oint\!\!\!\oint} \vec{A}(\vec{r})\bullet dS\hat{n}$$

since S is a closed surface, the unit normal $\hat{n}$ of dS (elemental area) is the <u>outward</u> normal

Conversion of a surface integral to a volume integral.

$$\iiint_{\text{volume region}} dV \left[\vec{\nabla} \bullet \vec{A}(\vec{r}) \right] = \oiint_{\text{surface enclosing that region}} \vec{A}(\vec{r}) \bullet dS\hat{n}$$

volume region

surface enclosing that region

Physical Meaning:

Integration of the faucets

(source of vector field) over a

volume

is equal to the

flux flowing out through the

surface enclosing the volume.

$$\operatorname{div}\vec{A} = \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] = \vec{\nabla} \bullet \vec{A}$$

How shall we express 'divergence' in cylindrical polar coordinate system?

$$\vec{\nabla} \bullet \vec{A} =$$

$$\left[\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}\right] \bullet \left[\hat{e}_\rho A_\rho(\rho,\varphi,z) + \hat{e}_\varphi A_\varphi(\rho,\varphi,z) + \hat{e}_z A_z(\rho,\varphi,z)\right]$$

There are TWO OPERATIONS here!

vector algebra

calculus take derivatives

$$\vec{\nabla} \bullet \vec{A} = \left[\hat{e}_\rho \frac{\partial}{\partial \rho}\right] \bullet \left[\hat{e}_\rho A_\rho(\rho,\varphi,z) + \hat{e}_\varphi A_\varphi(\rho,\varphi,z) + \hat{e}_z A_z(\rho,\varphi,z)\right] +$$

$$\left[\hat{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi}\right] \bullet \left[\hat{e}_\rho A_\rho(\rho,\varphi,z) + \hat{e}_\varphi A_\varphi(\rho,\varphi,z) + \hat{e}_z A_z(\rho,\varphi,z)\right] +$$

$$\left[\hat{e}_z \frac{\partial}{\partial z}\right] \bullet \left[\hat{e}_\rho A_\rho(\rho,\varphi,z) + \hat{e}_\varphi A_\varphi(\rho,\varphi,z) + \hat{e}_z A_z(\rho,\varphi,z)\right]$$

$$\vec{\nabla}\bullet\vec{A}=\left[\hat{e}_\rho\frac{\partial}{\partial\rho}\right]\bullet\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]+$$
$$\left[\hat{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right]\bullet\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]+$$
$$\left[\hat{e}_z\frac{\partial}{\partial z}\right]\bullet\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]$$

Note that the components A_ρ, A_φ, A_z each depends on (ρ,φ,z)

.... but the unit vectors $\hat{e}_\rho, \hat{e}_\varphi$ also depends on φ

$$\vec{\nabla}\bullet\vec{A}=\hat{e}_\rho\bullet\left\{\frac{\partial}{\partial\rho}\right\}\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]+$$
$$\hat{e}_\varphi\bullet\left\{\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right\}\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]+$$
$$\hat{e}_z\bullet\left\{\frac{\partial}{\partial z}\right\}\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]$$

$$\vec{\nabla}\bullet\vec{A}=\hat{e}_\rho\bullet\left\{\frac{\partial}{\partial\rho}\right\}\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]+$$

$$\hat{e}_\varphi\bullet\left\{\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right\}\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]+$$

$$\hat{e}_z\bullet\left\{\frac{\partial}{\partial z}\right\}\left[\hat{e}_\rho A_\rho(\rho,\varphi,z)+\hat{e}_\varphi A_\varphi(\rho,\varphi,z)+\hat{e}_z A_z(\rho,\varphi,z)\right]$$

$$\frac{\partial\hat{e}_\rho}{\partial\rho}=0,\ \frac{\partial\hat{e}_\rho}{\partial\varphi}=\hat{e}_\varphi,$$

$$\frac{\partial\hat{e}_\varphi}{\partial\rho}=0,\ \frac{\partial\hat{e}_\varphi}{\partial\varphi}=-\hat{e}_\rho$$

$$\vec{\nabla}\bullet\vec{A}=\frac{\partial}{\partial\rho}A_\rho(\rho,\varphi,z)+\frac{1}{\rho}A_\rho(\rho,\varphi,z)+$$

$$+\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_\varphi(\rho,\varphi,z)+\frac{\partial}{\partial z}A_z(\rho,\varphi,z)$$

Expression for 'divergence' in spherical polar coordinate system

$$\frac{\partial \hat{e}_r}{\partial r} = \vec{0}$$
$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$
$$\frac{\partial \hat{e}_r}{\partial \varphi} = \sin\theta \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = \vec{0}$$
$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$
$$\frac{\partial \hat{e}_\theta}{\partial \varphi} = \cos\theta \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\varphi}{\partial r} = \vec{0}$$
$$\frac{\partial \hat{e}_\varphi}{\partial \theta} = \vec{0}$$
$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\theta \hat{e}_\theta - \sin\theta \hat{e}_r$$

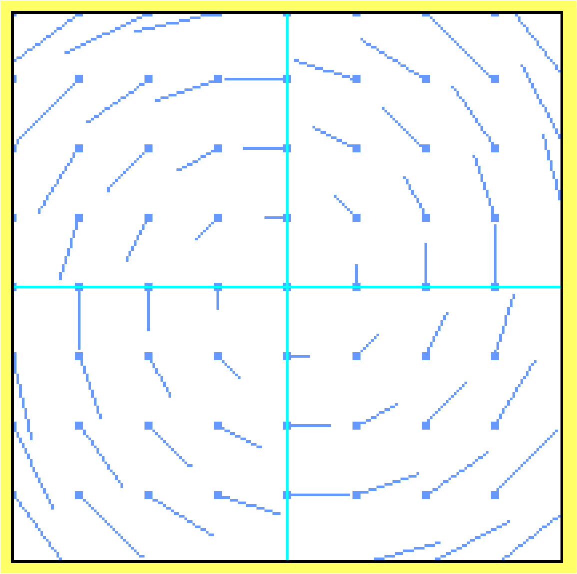
$$\vec{\nabla} \bullet \vec{A} =$$

$$\left\{ \hat{\mathrm{e}}_r \frac{\partial}{\partial r} + \hat{\mathrm{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathrm{e}}_\varphi \frac{1}{r\sin\theta} \frac{\partial}{\partial \varphi} \right\} \bullet \left[\hat{\mathrm{e}}_r A_r(r,\theta,\varphi) + \hat{\mathrm{e}}_\theta A_\theta(r,\theta,\varphi) + \hat{\mathrm{e}}_\varphi A_\varphi(r,\theta,\varphi) \right]$$

$$=$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 A_r(r,\theta,\varphi) \right] + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} \left[A_\theta(r,\theta,\varphi) \sin\theta \right] + \frac{1}{r\sin\theta} \frac{\partial}{\partial \varphi} A_\varphi(r,\theta,\varphi)$$

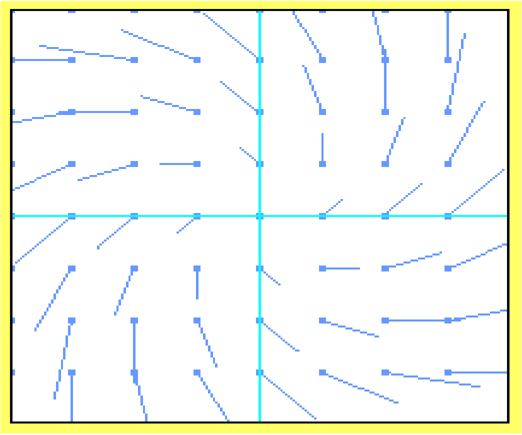
Examples for solenoidal and nonsolenoidal fields



$$\vec{\nabla}\bullet\vec{A}=0$$

Influx balances the outflux

Solenoidal ⟶ Example: $\vec{\mathbf{B}}$



$$\vec{A}=(x\text{-}y)\hat{e}_x+(x+y)\hat{e}_y$$

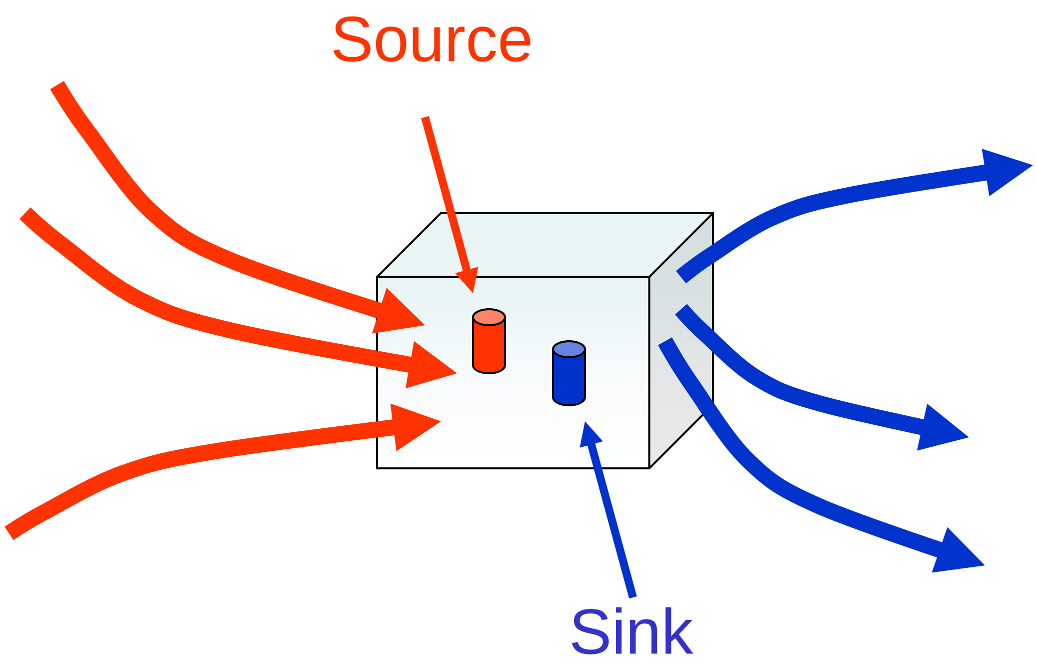
$$\vec{\nabla}\bullet\vec{A}=2$$

$$\operatorname{div}\vec{A}=\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]=\vec{\nabla}\bullet\vec{A}$$

Flux crossing a surface

$$= \iint_{\text{surface}} \vec{A}(\vec{r}) \bullet \overrightarrow{da}$$

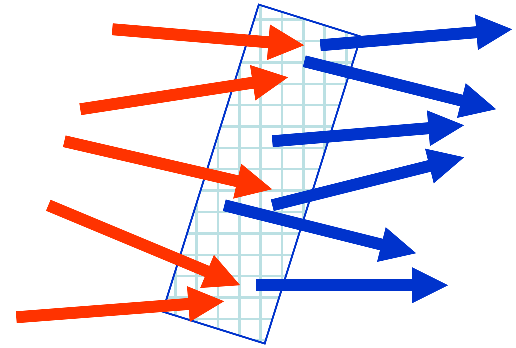
Is there any net accumulation of the flux in a volume element?



Sources and Sinks may be present in the region !

What happens when the size of the volume element shrinks, becoming infinitesimally small?

Consider a mass/charge density ρ_m or ρ_c crossing a certain cross-section of area at a certain rate.



$$\rho_m(\vec{r})\vec{\mathrm{v}}(\vec{r}) \text{ has the dimensions}$$

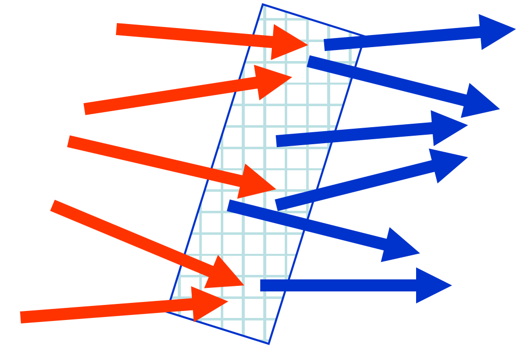
$$\left[\mathrm{ML^{-3}\ LT^{-1}}\right] = \mathrm{ML^{-2}\ T^{-1}}$$

$\rho_c(\vec{r})\vec{\mathrm{v}}(\vec{r})$ has the dimensions $\left[\mathrm{QL^{-3}\ LT^{-1}}\right] = \mathrm{QL^{-2}\ T^{-1}}$

Amount of charge crossing unit area in unit time

Amount of mass/charge crossing unit area in unit time

Consider a mass/charge density ρ_m or ρ_c crossing a certain cross-section of area at a certain rate.



Amount of mass/charge crossing unit area in unit time:

Physical quantity of interest: Density x Velocity

$$\vec{J}(\vec{r}) = \rho(\vec{r})\vec{\mathrm{v}}(\vec{r})$$

Current Density Vector

current crossing unit area

Mass/Charge Current Density Vector

Remember!

Sources/Sinks may be present in the region !

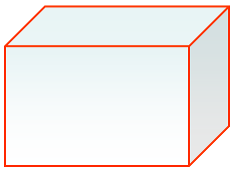
Divergence theorem: Conservation principle

Conservation of mass or charge

$\rho(\vec{r},t)$ represents mass/charge density

$\vec{J}(\vec{r},t)$: mass/charge current density

What shall we get if we integrate the flux emanating from all the six enclosing surfaces?



$$\iint_S \vec{J}(\vec{r})\bullet\hat{n}ds = I$$

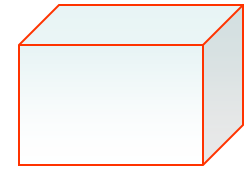
$$\underbrace{\iiint}_{\text{volume region}} dV\left[\vec{\nabla}\bullet\vec{J}(\vec{r})\right] = \underbrace{\oiint}_{\text{surface enclosing that region}} \vec{J}(\vec{r})\bullet dS\hat{n}$$

i.e. $\oiint_S \vec{J}(\vec{r}).\hat{n}ds = -\frac{\partial q_{total}}{\partial t} = -\frac{\partial}{\partial t}\iiint_V \rho dV = -\iiint_V \frac{\partial \rho}{\partial t} dV$

<u>Negative sign:</u> Net current oozing out of that region.

Outward flux is at the expense of the charge inside!

Divergence theorem: Conservation principle



$$\iiint dV \left\{ \vec{\nabla} \bullet \vec{J}(\vec{r}) \right\} = \oiint \vec{J}(\vec{r}) \bullet dS\hat{n}$$

volume region — surface enclosing that region

$$\iint_S \vec{J}(\vec{r}) \bullet \hat{n} dS = I$$

$$\text{i.e. } \oiint_S \vec{J}(\vec{r}) \bullet \hat{n} dS = -\frac{\partial q_{total}}{\partial t} = -\frac{\partial}{\partial t} \iiint_V \rho dV = \iiint_V \left\{ -\frac{\partial \rho}{\partial t} \right\} dV$$

Compare the integrands of the definite volume integrals

Integral and Differential forms of the equation of continuity: conservation principle

$$\vec{\nabla} \bullet \vec{J}(\vec{r}) = -\frac{\partial \rho}{\partial t}$$

$$\vec{\nabla} \bullet \vec{J}(\vec{r}) + \frac{\partial \rho}{\partial t} = 0$$

Equation of Continuity

Divergence theorem: Conservation principle

Equation of Continuity

$$\iiint_{\text{volume region}} dV \left\{ \vec{\nabla} \bullet \vec{J}(\vec{r}) \right\} = \oiint \vec{J}(\vec{r}) \bullet dS\hat{n}$$

$$\iiint_{\text{volume region}} dV \left\{ \vec{\nabla} \bullet \vec{J}(\vec{r}) + \frac{\partial \rho}{\partial t} \right\} = 0$$

$$\vec{\nabla} \bullet \vec{J}(\vec{r}) = -\frac{\partial \rho}{\partial t}$$

$$\vec{\nabla} \bullet \vec{J}(\vec{r}) + \frac{\partial \rho}{\partial t} = 0$$

Divergence theorem:

In the absence of the creation or destruction of matter (no 'source' or 'sink'), the density within a region of space can change only by having 'matter' flow into or out of the region through the surface that bounds it.

We shall take a break here…….

Questions ?

Comments ?

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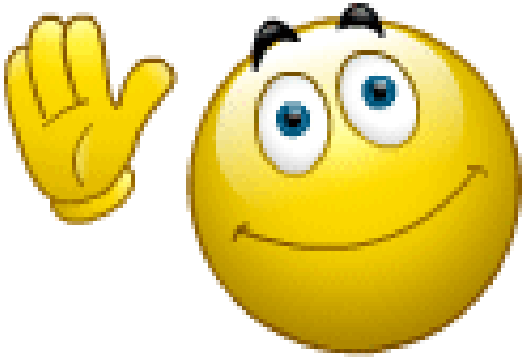
Next: L28

Equation of Fluid Motion

'continuum limit'

Lagrangian / Euler

description of fluid flow



http://www.physics.iitm.ac.in/~labs/amp/

STiCM

Select / Special Topics in Classical Mechanics

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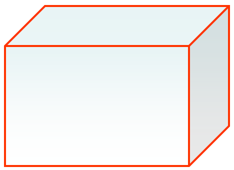
STiCM Lecture 28

Unit 8 : Gauss' Law; Equation of Fluid Motion

recapitulate from Lecture #27

Divergence theorem: Conservation principle



$$\iiint dV \left\{ \vec{\nabla} \bullet \vec{J}(\vec{r}) \right\} = \oiint \vec{J}(\vec{r}) \bullet dS\hat{n}$$

volume region — surface enclosing that region

$$\oiint_S \vec{J}(\vec{r}) \bullet \hat{n} dS = I$$

$$\text{i.e. } \oiint_S \vec{J}(\vec{r}) \bullet \hat{n} dS = -\frac{\partial q_{total}}{\partial t}$$

$$= -\frac{\partial}{\partial t} \iiint_V \rho dV = \iiint_V \left\{ -\frac{\partial \rho}{\partial t} \right\} dV$$

Compare the integrands of the definite volume integrals

Integral and Differential forms of the equation of continuity: conservation principle

$$\vec{\nabla} \bullet \vec{J}(\vec{r}) = -\frac{\partial \rho}{\partial t}$$

$$\vec{\nabla} \bullet \vec{J}(\vec{r}) + \frac{\partial \rho}{\partial t} = 0$$

Equation of Continuity

FLUID MECHANICS

We consider an incompressible fluid.

Under the application of a force, a solid gets pushed/pulled/spun…. or deformed.

A fluid ‘flows’

Deformation of solids, fluid flow: “rheology”
Non-Newtonian fluids --- example paints, foams, molten plastics…..

Ever walked on a fluid ?

http://www.youtube.com/watch?v=f2XQ97XHjVw

FLUID MECHANICS

Non-Newtonian fluids --- example

paints, foams, molten plastics…..

Matter behaves like a solid when pressure is exerted on it, and like a liquid when only little or no pressure is exerted on it.

The fluid's viscosity depends not on shear but on the rate of change of shear

--- complex systems;

but we shall work with “IDEAL” liquids

$$Density\ \rho(\vec{r}) = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}$$

$$What\ is\ the\ meaning\ of\ the\ \lim\ \delta V \to 0\,?$$

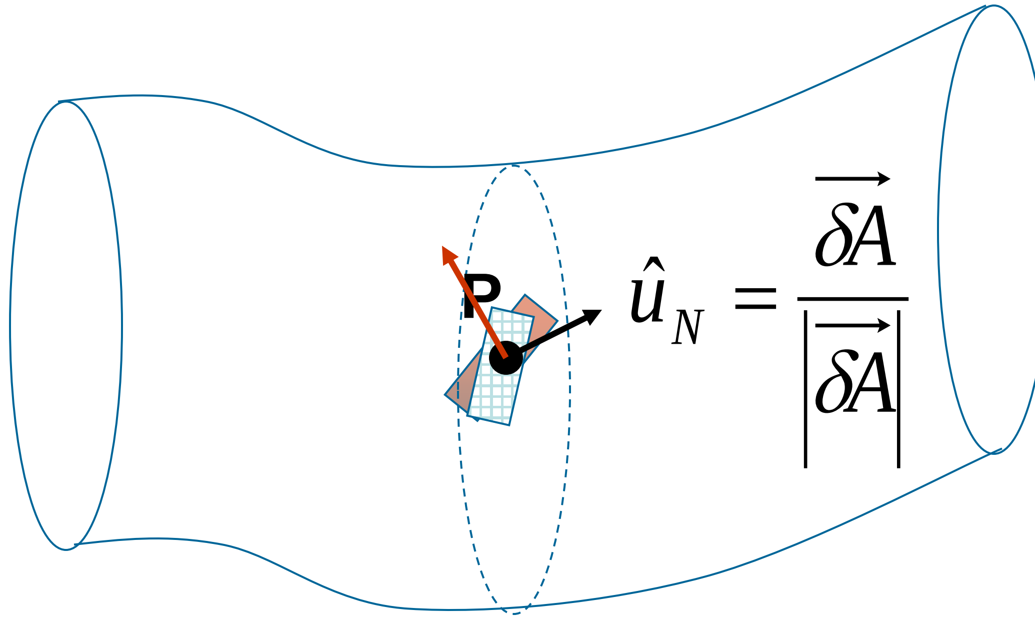
Classical fluid: continuum mechanics,
continuously divisible matter.

$\left|\vec{S}\right|$ does not depend on the direction of $\hat{u}_N$.

Pascal's law.

Stress at the point P is $\vec{S}$.

Blaise Pascal
1623-1662



The unit normal $\hat{u}_N$ can take any orientation.

"Let no one say that I have said nothing new... the arrangement of the subject is new. ...."

http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Pascal.html

$\left|\vec{S}\right|$ does not depend on the direction of $\hat{u}_N$.

Pascal's law.

The pressure is the same in every direction. The shape of the container does not matter.

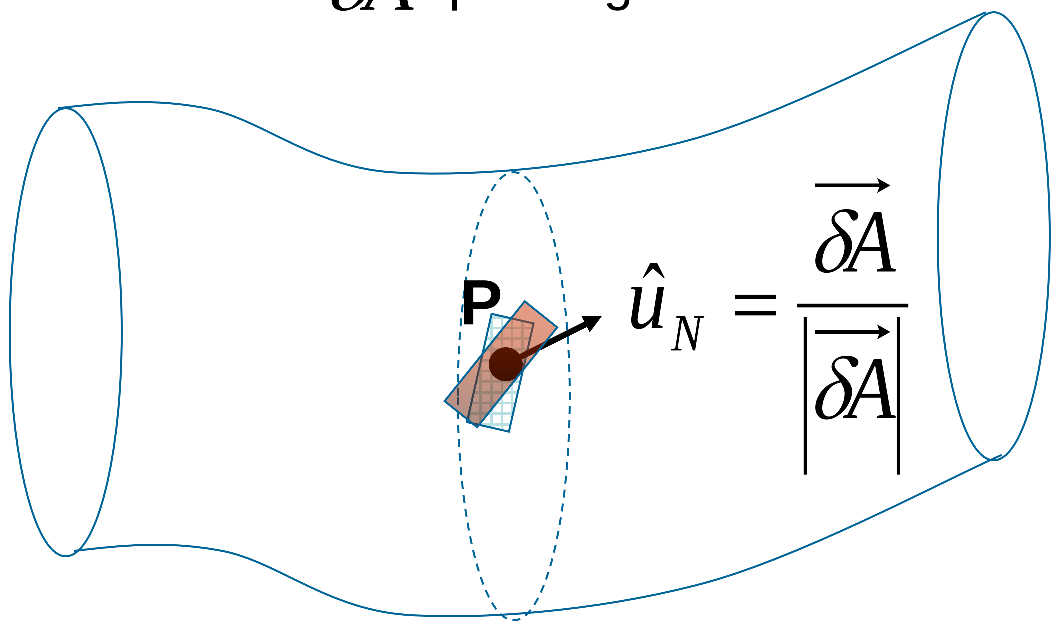
The pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the container.

some definitions.....

To understand the term ‘ideal’ fluid, we first define (i) ‘tension’, (ii) ‘compressions’ and (iii) ‘shear’.

Consider the force $\vec{F}$ on a tiny elemental area $\overrightarrow{\delta A}$ passing through point P in the liquid.

Stress at the point P is $\vec{S}$.



$$\hat{u}_N = \frac{\overrightarrow{\delta A}}{\left|\overrightarrow{\delta A}\right|}$$

$\vec{S} \bullet \hat{u}_N = \left|\vec{S}\right|$ → $\vec{S}$: **Tension**

$\vec{S} \bullet \hat{u}_N = 0$ → $\vec{S}$: **Shear**

$\vec{S} \bullet \hat{u}_N = -\left|\vec{S}\right|$ → $\vec{S}$: **Compression**

The unit normal $\hat{u}_N$ can take any orientation.

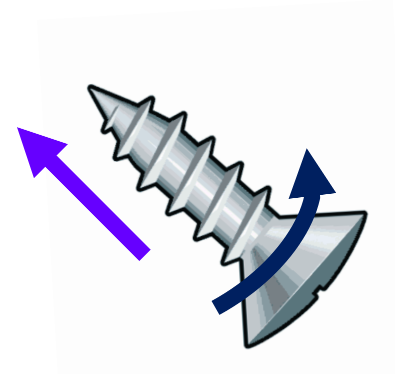
An ideal fluid is one in which stress at any point is essentially one of COMPRESSION.

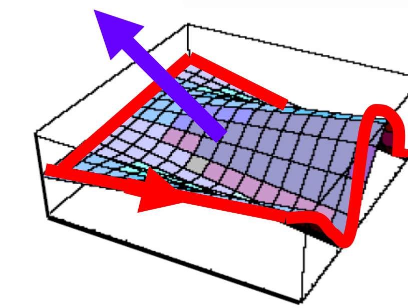
$$\vec{S} \bullet \hat{u}_N = \left|\vec{S}\right|$$ **Tension**

$$\vec{S} \bullet \hat{u}_N = 0$$ **Shear**

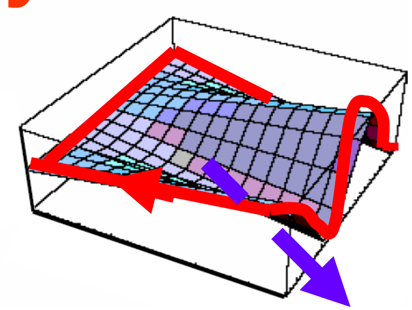
$$\vec{S} \bullet \hat{u}_N = -\left|\vec{S}\right|$$ **Compression**

Ideal fluid





C traversed one way



C traversed the other way

DIRECTION $\hat{u}_N$ of the 'ORIENTATION' OF THE SURFACE ELEMENT

The state of a fluid is completely determined by FIVE quantities:

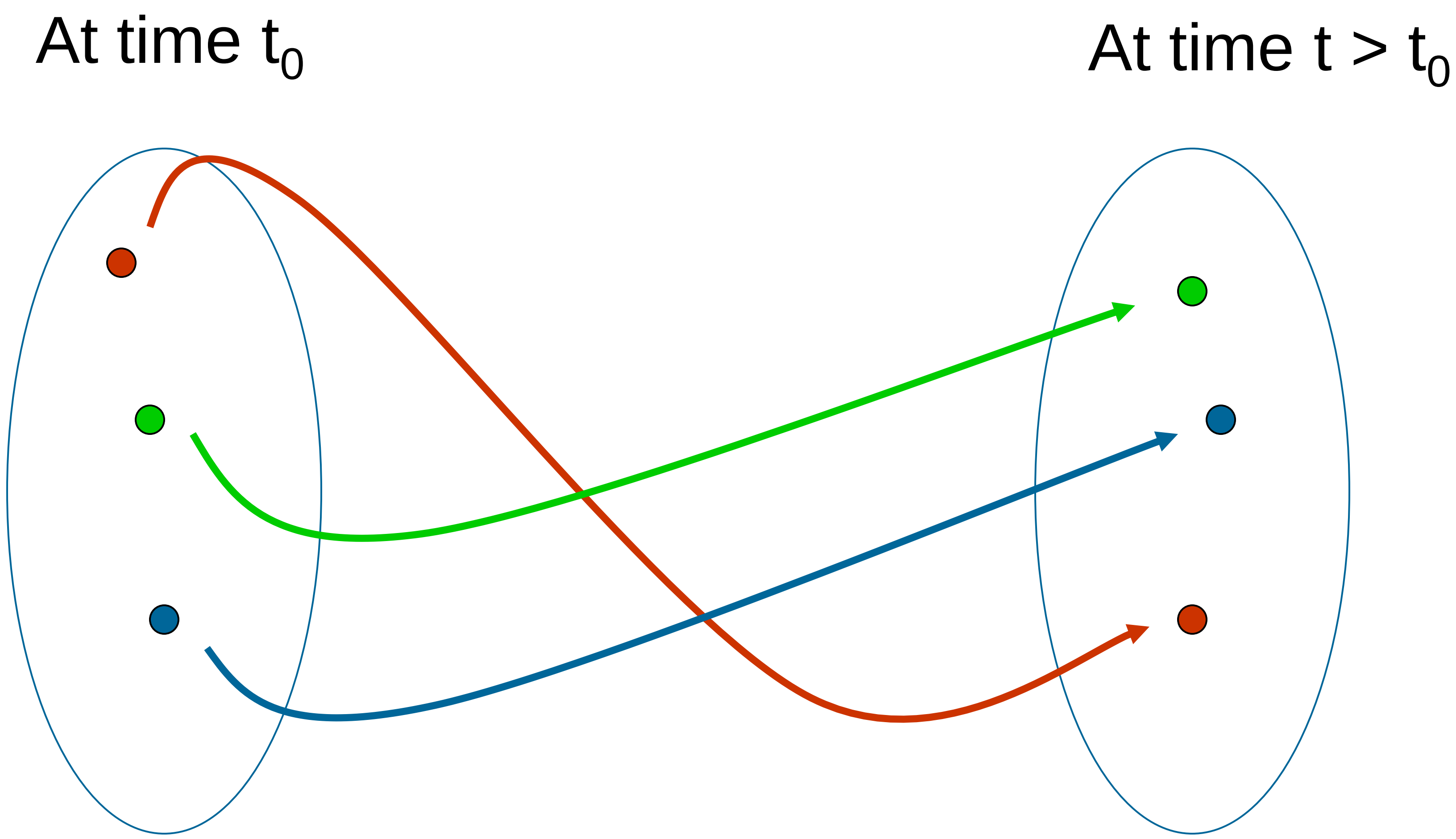
$$(1,2,3): \text{Three components of the velocity}$$

$$\text{at each point}: \vec{\mathrm{v}}(\vec{r}).$$

$$(4): \text{The pressure } \mathrm{p}(\vec{r},t).$$

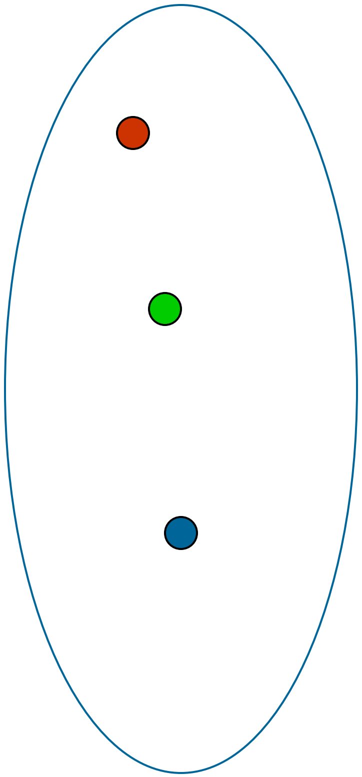
$$(5): \text{The density } \rho(\vec{r},t).$$

Above, we consider '**Eulerian**' position vector of a point with reference to a chosen frame of reference. It is not the position vector of any particular fluid molecule/particle.



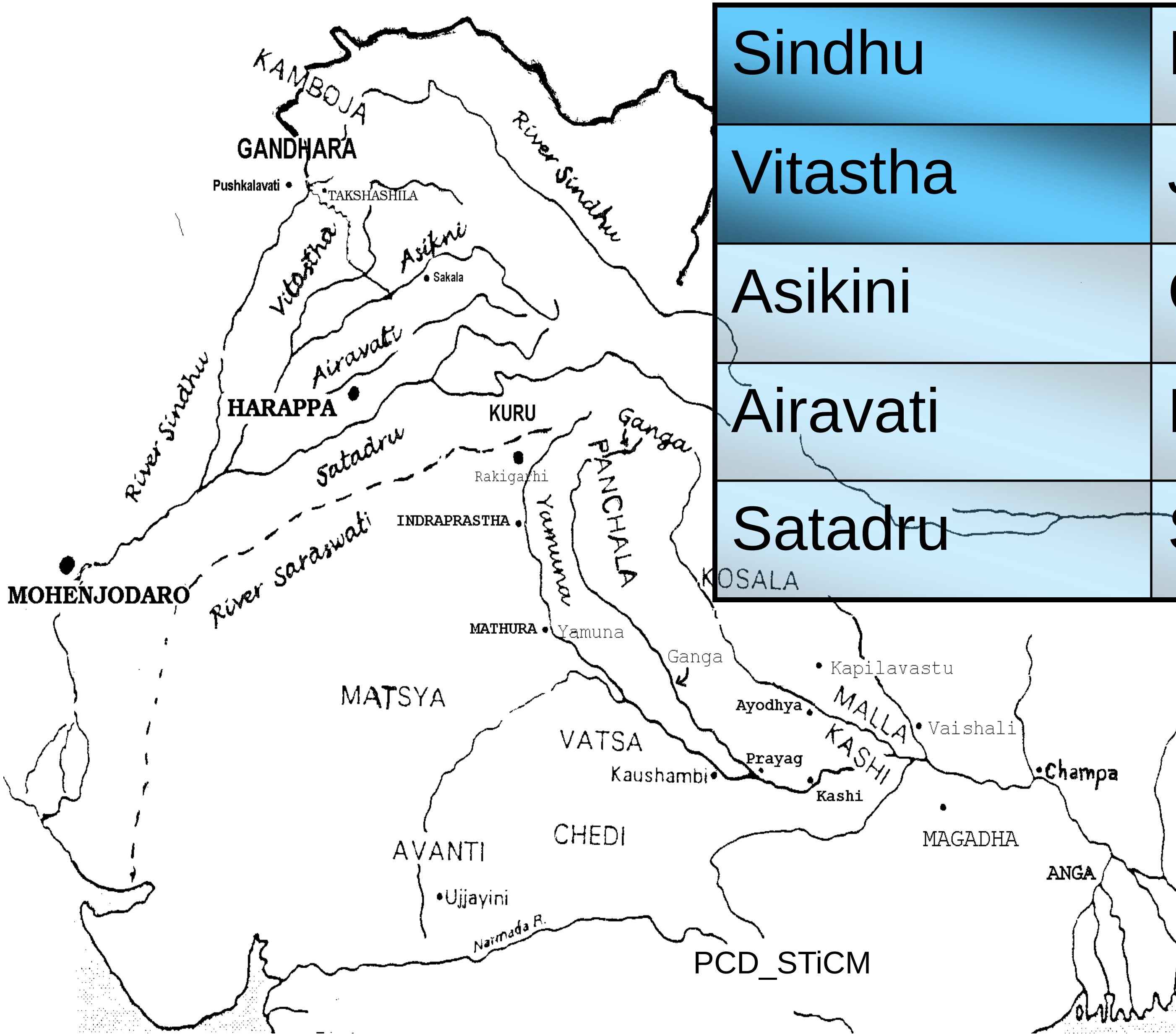
In the **Lagrangian** viewpoint, one tracks the **evolution in phase space of the entire continuous medium**; has huge amount of detailed information.

- not so in the Eulerian description.



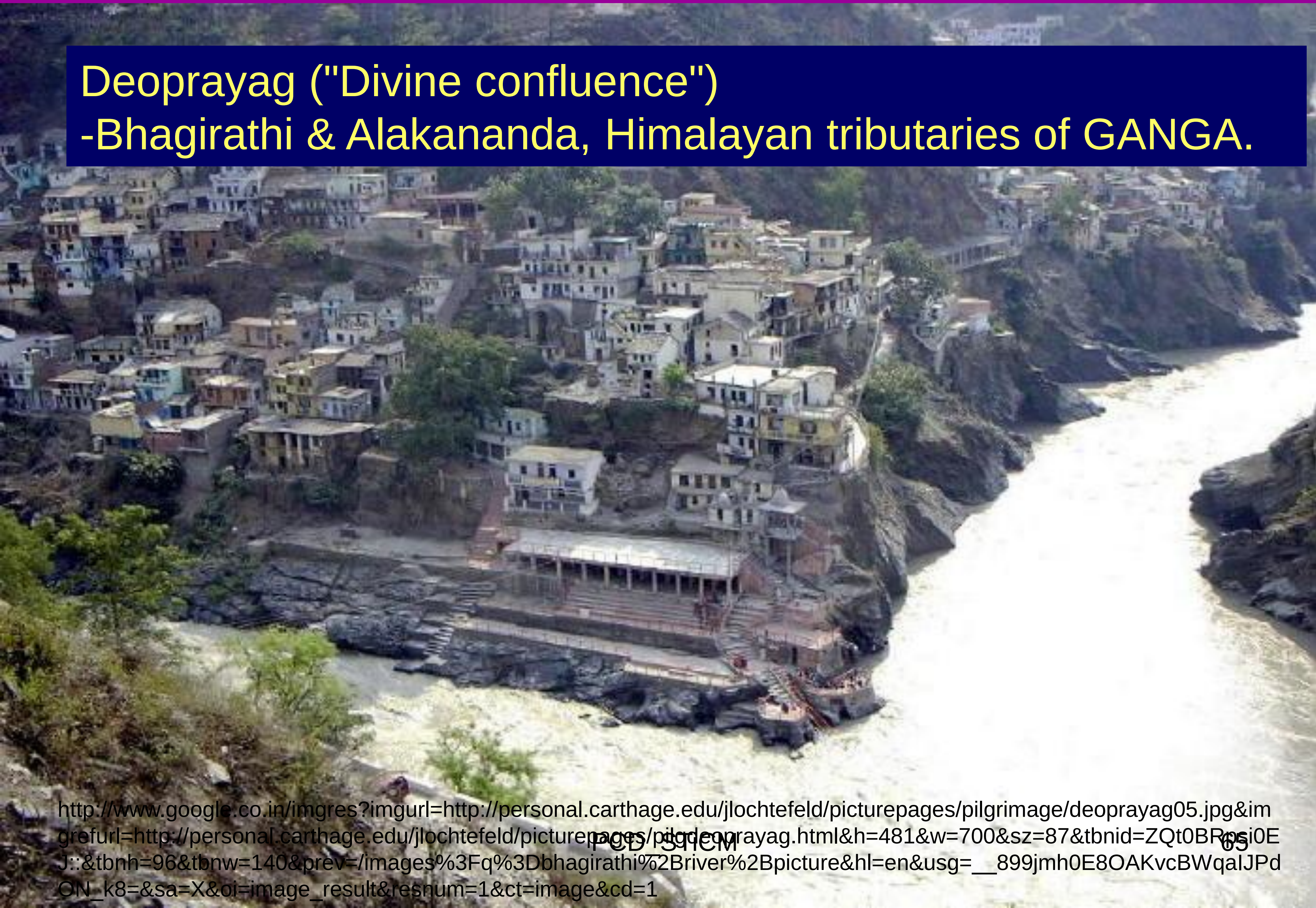
In the Eulerian description, one is interested in quantities such as the density $\rho(\vec{r})$ or the velocity $\vec{v}(\vec{r})$ and pressure $p(\vec{r})$ of an *arbitrary* fluid particle at $\vec{r}$.

How do we track density $\rho(\vec{r})$ & velocity $\vec{v}(\vec{r})$ at a point in a river?



Sindhu	Indus
Vitastha	Jhelum
Asikini	Chenab
Airavati	Ravi
Satadru	Sutlej

How do we track density $\rho(\vec{r})$ & velocity $\vec{v}(\vec{r})$ at a point in a river?



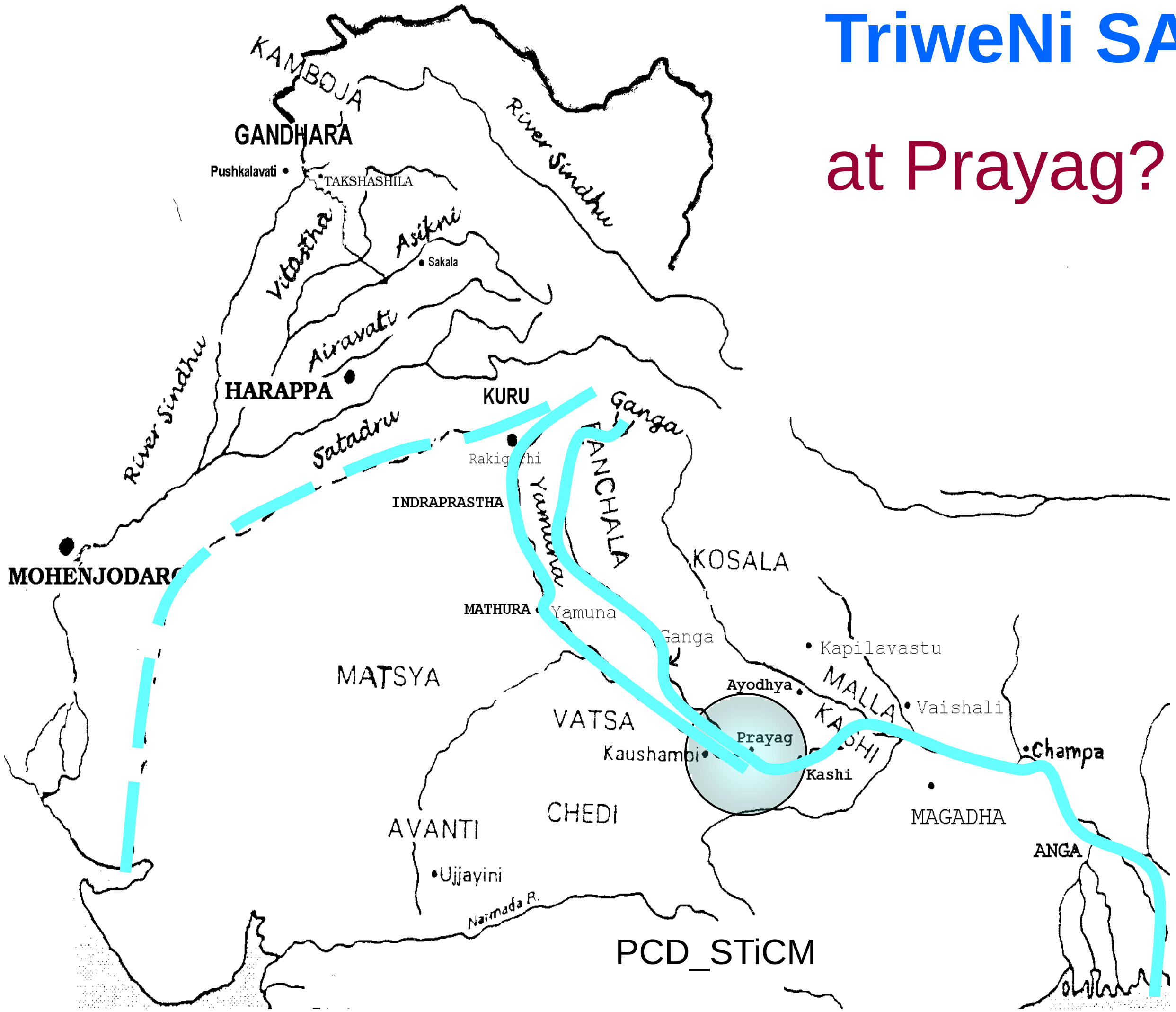
Deoprayag ("Divine confluence")
-Bhagirathi & Alakananda, Himalayan tributaries of GANGA.

http://www.google.co.in/imgres?imgurl=http://personal.carthage.edu/jlochtefeld/picturepages/pilgrimage/deoprayag05.jpg&imgrefurl=http://personal.carthage.edu/jlochtefeld/picturepages/pilgrimage/deoprayag.html&h=481&w=700&sz=87&tbnid=ZQt0BRpsi0EJ::&tbnh=96&tbnw=140&prev=/images%3Fq%3Dbhagirathi%2Briver%2Bpicture&hl=en&usg=__899jmh0E8OAKvcBWqaIJPdON_k8=&sa=X&oi=image_result&resnum=1&ct=image&cd=1

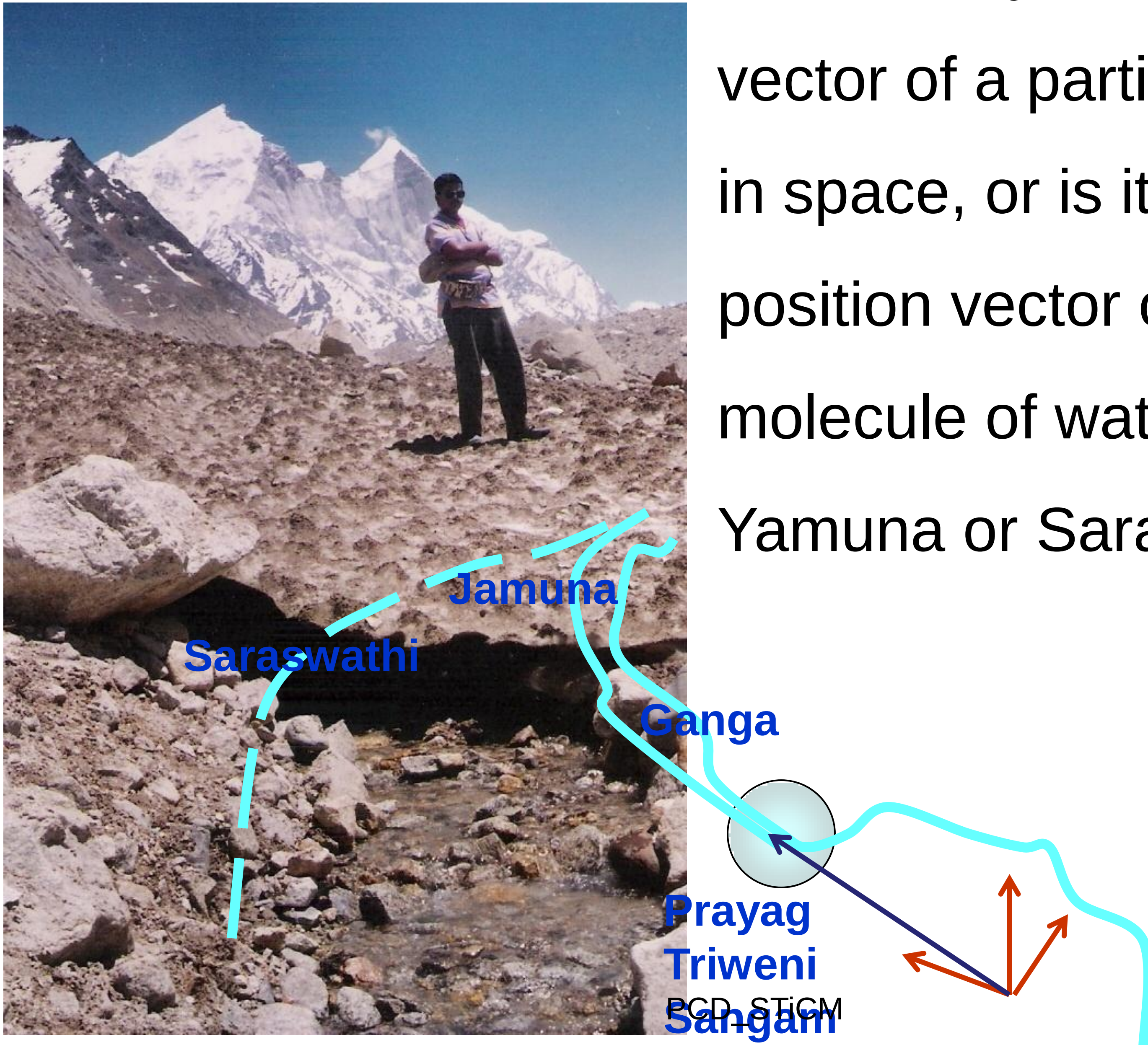
How do we have

TriweNi SANGAM

at Prayag?



Is ◯ merely the position vector of a particular point in space, or is it the position vector of a moving molecule of water Ganga, Yamuna or Saraswathi?



‘LAGRANGIAN’ TRACKING:
The waters of Yamuna would mix with the waters of Saraswathi and bring them to Prayag, into the Ganga!

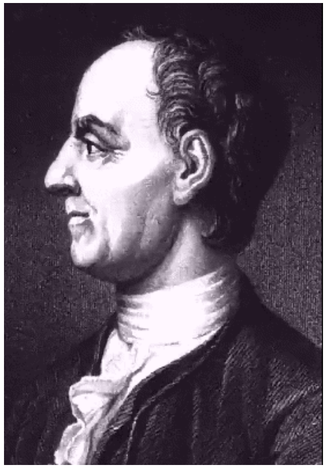
Equation of motion for fluids
two basic approaches

Lagrangian Approach:
Follow the motion of some particle of the fluid; this must be done for all particles of the fluid

Joseph-Louis Lagrange
1736 - 1813

Eulerian Approach:

Follow the velocity and density of fluid at a particular point; this must be done for all points in space



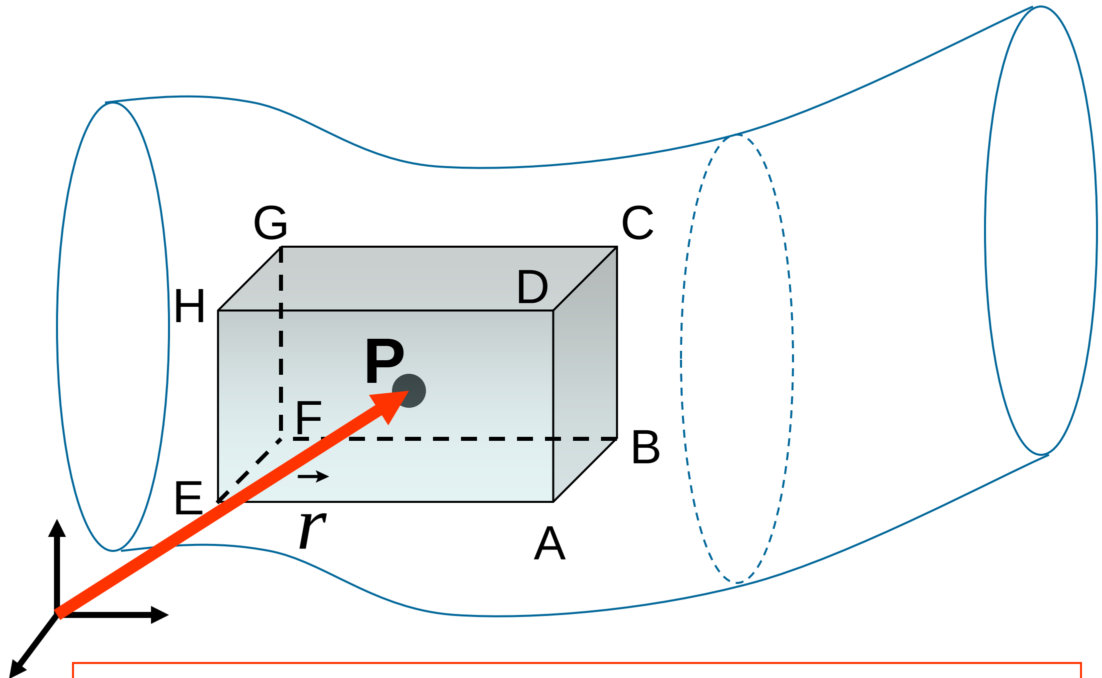
Leonhard Euler
(1707-1783)

Quantities of interest:

$$* \text{velocity} : \vec{v}(\vec{r}).$$

$$* \, pressure : \mathrm{p}(\vec{r}, t).$$

$$* \, density : \rho(\vec{r}, t).$$



Mass Current Density Vector

$$\vec{J}(\vec{r},t) = \rho(\vec{r},t)\vec{v}(\vec{r},t)$$

$$Dimensions : ML^{-2}T^{-1}$$

Measure of the amount of mass crossing unit area in unit time.

Amount of mass of fluid crossing face EFGH in unit time =

$$\lim_{\delta t \to 0} \frac{\delta m}{\delta t} = \lim_{\delta t \to 0} \frac{\rho \delta V}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\rho \delta x \delta y \delta z}{\delta t} = \rho \, v_x \, \delta y \delta z$$

$$= J_{x,EFGH} \delta y \delta z$$

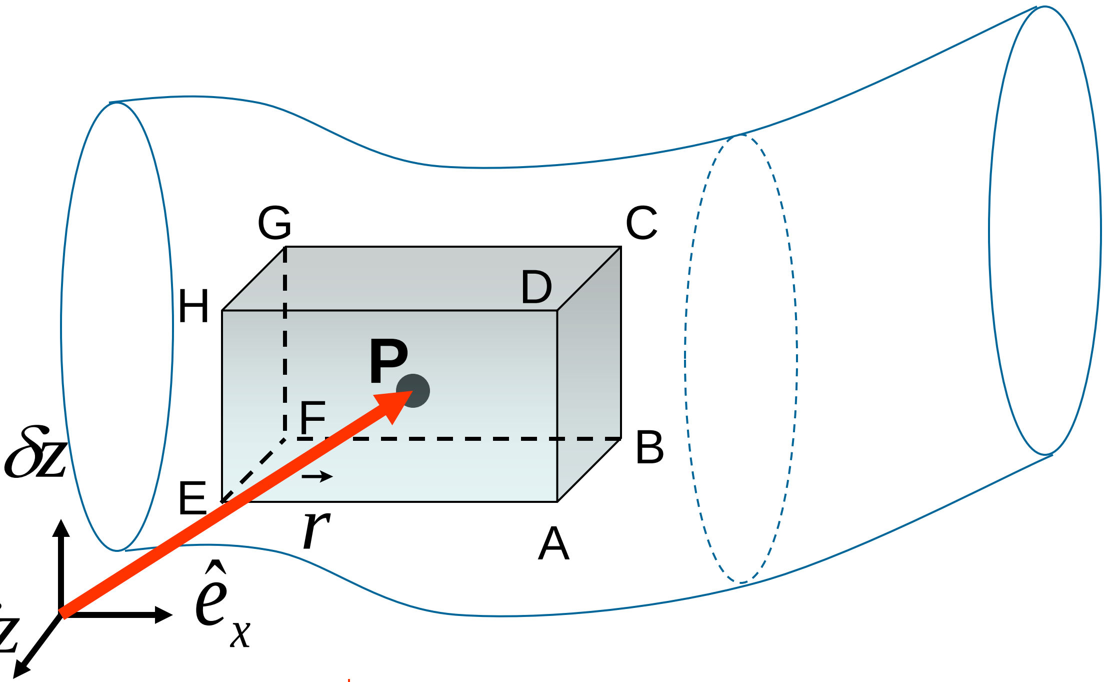
Amount of mass of fluid crossing face EFGH in unit time =

$$= \lim_{\delta t \to 0} \frac{\delta m}{\delta t} = \lim_{\delta t \to 0} \frac{\rho \delta V}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\rho \delta x \delta y \delta z}{\delta t} = \rho v_x \delta y \delta z$$

$$= \lim_{\delta t \to 0} \frac{\rho \delta x \delta y \delta z}{\delta t} = \rho v_x \delta y \delta z$$

$$= \left\{ J_x(\vec{r}) + \left[\frac{\partial J_x}{\partial x} \right]_P \left(- \frac{\delta x}{2} \right) \right\} \delta y \delta z$$



Amount of mass of fluid crossing face ABCD in unit time

$$= \left\{ J_x(\vec{r}) + \left[\frac{\partial J_x}{\partial x} \right]_P \left(\frac{\delta x}{2} \right) \right\} \delta y \delta z$$

Net *OUTWARD* flux through the **two faces orthogonal to x-axis** =

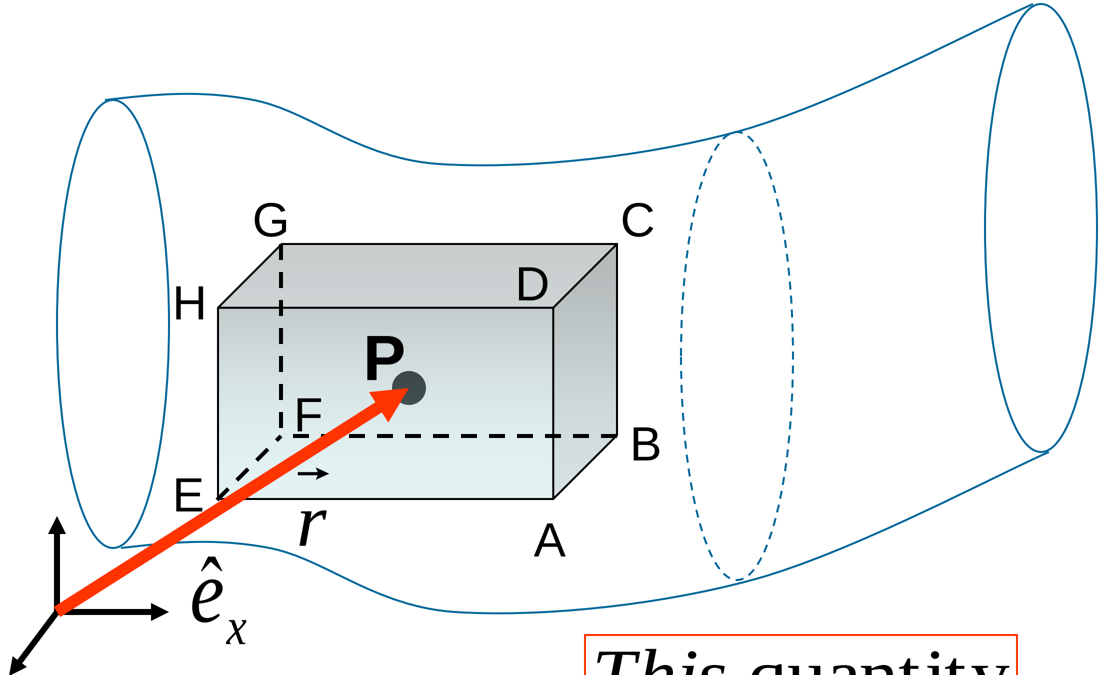
$$\left[\frac{\partial J_x}{\partial x} \right]_P \delta x \delta y \delta z$$

$$= \left[\frac{\partial J_x}{\partial x} \right]_P \delta V$$

Net *OUTWARD* flux **through the two faces orthogonal to x-axis** =

$$\left[\frac{\partial J_x}{\partial x}\right]_P \delta x\,\delta y\,\delta z$$

$$=\left[\frac{\partial J_x}{\partial x}\right]_P \delta V$$



Net *OUTWARD* flux **through the whole parallelepiped**, *per unit volume* =

$$\left\{\left[\frac{\partial J_x}{\partial x}\right]_P+\left[\frac{\partial J_y}{\partial y}\right]_P+\left[\frac{\partial J_z}{\partial z}\right]_P\right\}=\left[\vec{\nabla}\bullet\vec{J}\right]_P$$

This quantity must be equal to

$$-\frac{\partial\rho}{\partial t}$$

$$\vec{\nabla}\bullet\vec{J}=-\frac{\partial\rho}{\partial t}$$

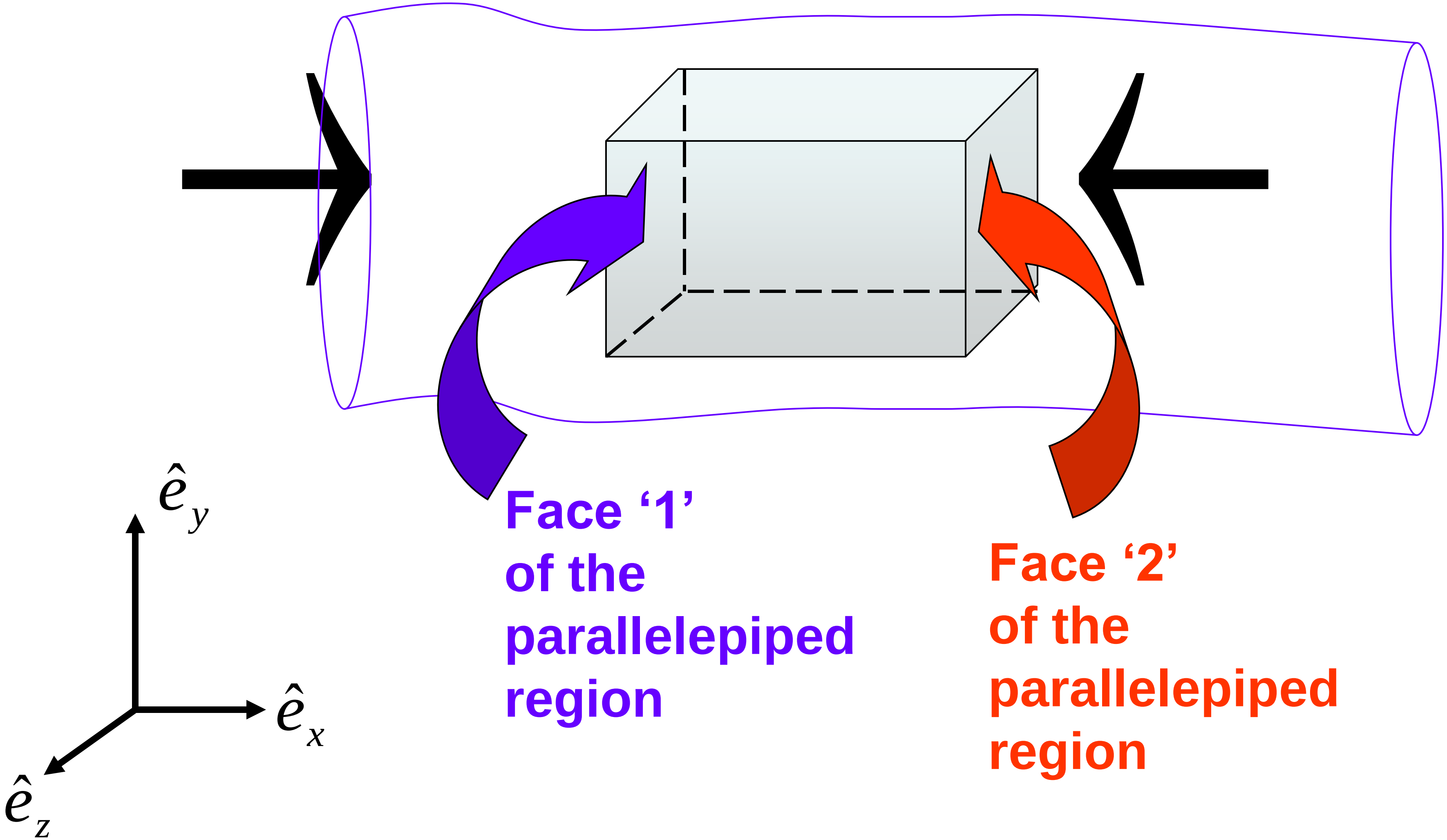
The choice of the term 'DIVERGENCE' is thus well justified.

Equation of Continuity
Conservation of Matter

$$\vec{\nabla} \bullet \vec{J}(\vec{r},t) = -\frac{\partial \rho(\vec{r},t)}{\partial t}$$

$$\vec{\nabla} \bullet \vec{J}(\vec{r},t) + \frac{\partial \rho(\vec{r},t)}{\partial t} = 0$$

$$\vec{J}(\vec{r},t) \bullet \hat{u} = \text{mass flux in the direction of } \hat{u}$$



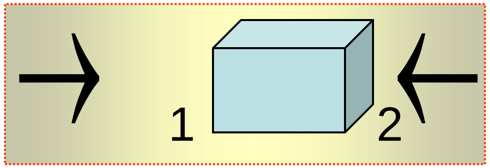
Ideal fluid: stress at any point is essentially one of COMPRESSION.

We can now develop the

EQUATION of MOTION

for a Newtonian Fluid

$$\vec{F}(1) = \left\{ p(\vec{r}) \ominus \left[\frac{\partial p}{\partial x}\right]_P \frac{\delta x}{2} \right\} (\delta y \delta z)(\oplus \hat{e}_x)$$



$\hat{e}_x \longrightarrow$

$$\vec{F}(2) = \left\{ p(\vec{r}) \oplus \left[\frac{\partial p}{\partial x}\right]_P \frac{\delta x}{2} \right\} (\delta y \delta z)(\ominus \hat{e}_x)$$

Ideal fluid: stress at any point is essentially one of COMPRESSION.

$$\vec{F}(1) + \vec{F}(2) = -\left[\frac{\partial p}{\partial x}\right]_P \delta x (\delta y \delta z)(+\hat{e}_x) = -\left[\frac{\partial p}{\partial x}\right]_P \delta V \, \hat{e}_x$$

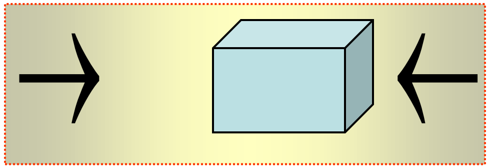
$$\sum_{i=1}^{6} \vec{F}(i) = -\vec{\nabla} p \, \boxed{\delta V}$$

"HYDROSTATIC force"
There may be some additional external force acting, such as gravity.

$$\text{Net } \boxed{\text{force}} \text{ per unit volume} = -\vec{\nabla} p$$

= negative gradient of pressure

Net HYDROSTATIC force acting on the parallelepiped per unit volume $= -\vec{\nabla} p$



External force (such as gravity) acting on the parallelepiped per unit volume

$$= \lim_{\delta V \to 0} \frac{\vec{F}_{external}}{\delta V}$$

$$= \lim_{\delta V \to 0} \frac{\vec{F}_{external}}{\delta m} \frac{\delta m}{\delta V}$$

$$= \lim_{\delta V \to 0} \frac{\vec{F}_{external}}{\delta m} \rho(\vec{r})$$

$$= \vec{g} \rho(\vec{r})$$

Total **{hydrostatic + external (gravity)}** force acting on the parallelepiped per unit volume

$$\lim_{\delta V \to 0} \frac{\delta m}{\delta V} \frac{d\vec{v}}{dt}$$

$$\rho(\vec{r}) \frac{d\vec{v}}{dt} = -\vec{\nabla} p + \vec{g} \rho(\vec{r})$$

Mass x Acceleration

"Cause-Effect"

Newton's law:

Equation of Motion

Mass x Acceleration / "Cause-Effect"
Newton's law: Equation of Motion

$$\lim_{\delta V \to 0} \frac{\delta m}{\delta V} \frac{d\vec{v}}{dt}$$

$$\rho(\vec{r})\frac{d\vec{v}}{dt} = -\vec{\nabla} p + \vec{g}\rho(\vec{r})$$

$\vec{r}$: 'LAGRANGIAN' position vector of a moving/flowing fluid 'particle/molecule', not the EULERIAN position vector of a fixed point in space.

$$\vec{r}_{Lagrangian} = \vec{r}(t)$$ ⟸ This is a function of time

$\vec{r}_{Euler}$ ⟸ Fixed point in space, *not* a function of time

$\frac{d\vec{v}}{dt}$ Is the ACCELERATION of actual material/fluid particle/molecule, and not just the rate at which velocity of the fluid is changing at a fixed point in space.

Mass x Acceleration / "Cause-Effect"

Newton's law: Equation of Motion

$$\rho(\vec{r})\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \vec{g}\rho(\vec{r})$$

$$\frac{d\vec{v}}{dt} = \left[\frac{d}{dt}\right]\vec{v}(\vec{r}(t),t) = \left[\frac{d}{dt}\right]\vec{v}\left(x(t),y(t),z(t),t\right)$$

$$= \frac{\partial\vec{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{v}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{v}}{\partial z}\frac{dz}{dt} + \frac{\partial\vec{v}}{\partial t}$$

$$\frac{d\vec{v}}{dt} = \frac{dx}{dt}\frac{\partial\vec{v}}{\partial x} + \frac{dy}{dt}\frac{\partial\vec{v}}{\partial y} + \frac{dz}{dt}\frac{\partial\vec{v}}{\partial z} + \frac{\partial\vec{v}}{\partial t}$$

$$= \left[\vec{v}\bullet\vec{\nabla} + \frac{\partial}{\partial t}\right]\vec{v}$$

"CONVECTIVE DERIVATIVE OPERATOR"

$$i.e. \quad \frac{d}{dt} \equiv \left[\vec{v}\bullet\vec{\nabla} + \frac{\partial}{\partial t}\right]$$

The term 'convection' is a reminder of the fact that in the convection process, the transport of a material particle is involved.

Mass x Acceleration / "Cause-Effect"

Newton's law: Equation of Motion

$$\rho(\vec{r})\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \vec{g}\rho(\vec{r})$$

$$\frac{d\vec{v}(\vec{r},t)}{dt} = \frac{-\vec{\nabla}p}{\rho(\vec{r})} + \vec{g}_{external} = \frac{-\vec{\nabla}p}{\rho(\vec{r})} - \vec{\nabla}\phi$$

External force field, which we considered to be gravity

$$\left[\vec{v}\bullet\vec{\nabla} + \frac{\partial}{\partial t}\right]\vec{v}(\vec{r},t) = \frac{d\vec{v}(\vec{r},t)}{dt} = \frac{-\vec{\nabla}p}{\rho(\vec{r})} - \vec{\nabla}\phi + \vec{F}_{viscous}$$

Hydrodynamic term

Viscous, frictional, dissipative term. This terms makes "dry water wet"

- Feynman

We shall take a break here…….

Questions ?

Comments ?

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Next: L29

Unit 9 – Fluid Flow / Bernoulli's principle

….. but *which* Bernoulli ?